ON MINAKSHISUNDARAM-PLEIJEL ZETA FUNCTIONS OF SPHERES

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ABSTRACT. The aim of this paper is to show that the Minakshisundaram-Pleijel zeta function $Z_k(s)$ of k -dimensional sphere S^k , $k \geq 2$ (defined in $\Re(s) > \frac{k}{2}$ by

$$Z_k(s) = \sum_{n=1}^{\infty} \frac{P_k(n)}{[n(n+k-1)]^s}$$

with $(k-1)!P_k(n) = \mathcal{R}(n+1, k-2)(2n+k-1)$ where the “rising factorial” $\mathcal{R}(x, n) = x(x+1)\cdots(x+n-1)$ is defined for real number x and n non-negative integer) can be put in the form

$$(k-1)!Z_k(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{k-1}{2}\right)^{2l} \binom{-s}{l} \sum_{j=0}^{k-1} B_{k,j} \zeta(2s+2l-j, \frac{k+1}{2})$$

where $B_{k,j}$ are explicitly computed. The above formula allows us to find explicitly the residue of $Z_k(s)$ at the pole $s = \frac{k}{2} - n$, $n \in \mathbb{N}$,

$$\frac{1}{(k-1)!} \sum_{h=0}^{\frac{k}{2}-1} \sum_{\substack{l+h=n \\ l \geq 0}} (-1)^l \left(\frac{k-1}{2}\right)^{2l} \binom{n-\frac{k}{2}}{l} B_{k,k-2h-1}.$$

In passing, we also obtain apparently new relations among the Stirling numbers.

1. INTRODUCTION AND NOTATION

In 1949, Minakshisundaram and Pleijel introduced a generalized Dirichlet series associated with a compact riemannian manifold ([5]). It is a standard result that the eigenvalues of the Laplace operator acting on $L^2(M)$ form a sequence

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$$

where every λ_i is repeated with its multiplicity which is finite. The Minakshisundaram-Pleijel zeta function of the riemannian manifold M is defined in

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$\Re(s) > \frac{\dim M}{2}$ by

$$\zeta_M(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

This function is closely connected with the problem of the asymptotic behaviour of the eigenvalues. Minakshisundaram and Pleijel proved the meromorphic continuation of $\zeta_M(s)$ by using tools of global analysis; more precisely they proved that $\zeta_M(s)$ is analytic on $\mathbb{C}$ except at simple poles $s = \frac{\dim M}{2} - k$, $k \in \mathbb{N}$, and expressed the residues in terms of metric invariants of M in a quite involved manner so that they are not at all efficiently computable even if the eigenvalues are explicitly known (f.i. k -dimensional sphere or the projective real and complex spaces equipped with the standard metric) (see [1] and [3]).

In this note we express $\zeta_M(s)$, when $M = S^k$ or $\mathbb{P}^k(\mathbb{R})$, in a convenient way, which allows us to write down the residues explicitly.

The Minakshisundaram-Pleijel zeta function $Z_k(s)$ of k -dimensional sphere S^k takes the form, in $\Re(s) > \frac{k}{2}$

$$Z_k(s) = \sum_{n=1}^{\infty} \frac{P_k(n)}{[n(n+k-1)]^s}$$

with $(k-1)!P_k(n) = \mathcal{R}(n+1, k-2)(2n+k-1)$ (see [1]) where the “rising factorial” $\mathcal{R}(x, n) = x(x+1) \cdots (x+n-1)$ is defined for real number x and n non-negative integer. We observe that if $k=1$, the Minakshisundaram-Pleijel zeta function reduces to $2\zeta(2s)$ where $\zeta(s)$ is the Riemann zeta function.

2. STIRLING NUMBERS

We define the *Stirling numbers of the first kind* $s_{n,k}$ by

$$\mathcal{R}(x, n) = \sum_{k=0}^n (-1)^{n+k} s_{n,k} x^k.$$

We list some useful formulas concerning Stirling numbers here (see [4], Chapter IV, where the Stirling numbers are denoted by S_n^k)

$$(1a) \quad s_{n+1,k} = s_{n,k-1} - ns_{n,k} \quad (\text{see [4], 5 of p. 143}),$$

$$(1b) \quad s_{n,k} = \sum_{j=0}^n \binom{j}{k} s_{n+1,j+1} \quad (\text{see [4], 4. of p. 187}),$$

$$(1c) \quad (-1)^k s_{n+1,k+1} = \sum_{j=0}^n (-1)^j \binom{j}{k} s_{n,j} \quad (\text{see [4], 3. of p. 186}),$$

$$(1d) \quad s_{n+1,k} = \sum_{j=0}^n (-1)^j \frac{n!}{(n-j)!} s_{n-j,k-1}.$$

If we put

$$(2) \quad (k-1)!P_k(n) = \sum_{m=0}^{k-1} \alpha_{k,m} n^m,$$

we also have

$$(3) \quad \alpha_{k,m} = (-1)^{m+k+1}(s_{k,m+1} + s_{k-1,m}), \quad 0 \leq m \leq k-1.$$

Equation (3) follows from (1a) and (1c).

3. STATEMENT OF RESULTS

We get the following results

Theorem 1. *Let us consider*

$$L_k(s) = \sum_{n=1}^{\infty} \frac{P_k(n)}{[(n+\delta_1)(n+\delta_2)]^s}$$

where $\delta_1 \leq \delta_2$ are non-negative real numbers with $\delta_1 + \delta_2 = k-1$ and $P_k(n)$ is as above. If $k \geq 2$, then

$$(k-1)!L_k(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{\delta_2 - \delta_1}{2} \right)^{2l} \binom{-s}{l} \sum_{j=0}^{k-1} B_{k,j} \zeta(2s+2l-j, \frac{k+1}{2})$$

where

$$B_{k,j} = \sum_{p=0}^{k-j-1} (-1)^{k+j+1} \binom{j+p}{j} \left(\frac{k-1}{2} \right)^p (s_{k,j+p+1} + s_{k-1,j+p}), \quad 0 \leq j \leq k-1,$$

and $\zeta(s, \alpha)$ is the familiar Hurwitz zeta-function $\sum_{r=0}^{\infty} (r+\alpha)^{-s}$ with $\alpha \in \mathbb{R}$, $\alpha > 0$.

Corollary. Taking $\delta_1 = 0$, $\delta_2 = k-1$ we get for $Z_k(s)$

$$(k-1)!Z_k(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{k-1}{2} \right)^{2l} \binom{-s}{l} \sum_{j=0}^{k-1} B_{k,j} \zeta(2s+2l-j, \frac{k+1}{2}).$$

Theorem 2. *The residue of the Minakshisundaram-Pleijel zeta function $Z_k(s)$ at the pole $s = \frac{k}{2} - n$, $n \in \mathbb{N}$, is*

$$\frac{1}{(k-1)!} \sum_{h=0}^{\frac{k}{2}-1} \sum_{\substack{l+h=n \\ l \geq 0}} (-1)^l \left(\frac{k-1}{2} \right)^{2l} \binom{n - \frac{k}{2}}{l} B_{k,k-2h-1}.$$

4. PROOF OF THEOREM 1

Applying (5) of [2] to $L_k(s)$ we have

$$\begin{aligned}
 (k-1)!L_k(s) &= \sum_{l=0}^{\infty} (-1)^l \left(\frac{\delta_2 - \delta_1}{2} \right)^{2l} \binom{-s}{l} \\
 &\cdot \left[\alpha_{k,0} \zeta(2s+2l, \frac{k+1}{2}) \right. \\
 &\quad + \sum_{m=1}^{k-1} \sum_{j=0}^m \sum_{p=1}^m \alpha_{k,m} a_p^m P_{m-j}^m (m+1-p - \frac{k-1}{2}) \\
 &\quad \left. \cdot \zeta(2s+2l-j, p + \frac{k-1}{2}) \right]
 \end{aligned}
 \tag{4}$$

where, according the notation of [2]

$$\begin{aligned}
 a_p^m &= \sum_{t=1}^p (-1)^{p-t} \binom{m+1}{p-t} t^m, \\
 P_{m-j}^m(z) &= \frac{1}{m!} \sum_{h=1}^{m+1} s_{m+1,h} \binom{h-1}{j} z^{h-j-1}.
 \end{aligned}$$

If $k = 2$, we get

$$L_2(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{\delta_2 - \delta_1}{2} \right)^{2l} \binom{-s}{l} (2\zeta(2s+2l-1, \frac{3}{2})).$$

Let us assume that $k \geq 3$. First we reduce the parameters $p + \frac{k-1}{2}$ of the Hurwitz zeta functions $\zeta(2s+2l-j, p + \frac{k-1}{2})$ to $\frac{k+1}{2}$, so we have

$$\begin{aligned}
 (k-1)!L_k(s) &= \sum_{l=0}^{\infty} (-1)^l \left(\frac{\delta_2 - \delta_1}{2} \right)^{2l} \binom{-s}{l} \\
 &\cdot \left[\alpha_{k,0} \zeta(2s+2l, \frac{k+1}{2}) \right. \\
 &\quad + \sum_{m=1}^{k-1} \sum_{j=0}^m \alpha_{k,m} a_1^m P_{m-j}^m (m - \frac{k-1}{2}) \zeta(2s+2l-j, \frac{k+1}{2}) \\
 &\quad + \sum_{m=2}^{k-1} \sum_{j=0}^m \sum_{p=2}^m \alpha_{k,m} a_p^m P_{m-j}^m (m+1-p - \frac{k-1}{2}) \\
 &\quad \left. \cdot \left[\zeta(2s+2l-j, \frac{k+1}{2}) - \sum_{t=0}^{p-2} \left(t + \frac{k-1}{2} \right)^{j-2s-2l} \right] \right].
 \end{aligned}
 \tag{5}$$

We will prove that

$$(6) \quad \sum_{m=2}^{k-1} \sum_{j=0}^m \sum_{p=2}^m \sum_{t=0}^{p-2} \alpha_{k,m} a_p^m P_{m-j}^m \left(m+1-p-\frac{k-1}{2}\right) \left(t+\frac{k-1}{2}\right)^{j-2s-2l} = 0.$$

We rearrange the sums in left side of (6) as follows:

$$\sum_{t=0}^{k-3} \sum_{m=t+2}^{k-1} \sum_{p=t+2}^m \sum_{j=0}^m \alpha_{k,m} a_p^m P_{m-j}^m \left(m+1-p-\frac{k-1}{2}\right) \left(t+\frac{k-1}{2}\right)^{j-2s-2l} = 0.$$

Now, if $0 \leq t \leq k-3$ we obtain

$$\begin{aligned} & \sum_{m=t+2}^{k-1} \sum_{p=t+2}^m \sum_{j=0}^m \alpha_{k,m} a_p^m \frac{1}{m!} \sum_{h=1}^{m+1} s_{m+1,h} \binom{h-1}{j} \\ & \quad \cdot \left(m+1-p-\frac{k-1}{2}\right)^{h-1-j} \left(t+\frac{k-1}{2}\right)^j \\ &= \sum_{m=t+2}^{k-1} \sum_{p=t+2}^m \sum_{h=1}^{m+1} \frac{1}{m!} \alpha_{k,m} a_p^m s_{m+1,h} \\ & \quad \cdot \sum_{j=0}^m \binom{h-1}{j} \left(m+1-p-\frac{k-1}{2}\right)^{h-1-j} \left(t+\frac{k-1}{2}\right)^j \\ &= \sum_{m=t+2}^{k-1} \sum_{p=t+2}^m \sum_{h=1}^{m+1} \frac{1}{m!} \alpha_{k,m} a_p^m s_{m+1,h} (m+1-p+t)^{h-1} \\ &= \sum_{m=t+2}^{k-1} \sum_{p=t+2}^m \frac{1}{m!} \alpha_{k,m} a_p^m \frac{1}{(m+1-p+t)} \\ & \quad \cdot \sum_{h=1}^{m+1} (-1)^h s_{m+1,h} (p-m-t-1)^h = 0 \end{aligned}$$

since

$$\sum_{h=1}^{m+1} (-1)^h s_{m+1,h} (p-m-t-1)^h = \mathcal{R}(p-t-m-1, m+1) = 0.$$

So (6) is proved.

Now we can write

(7)

$$(k-1)! L_k(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{\delta_2 - \delta_1}{2}\right)^{2l} \binom{-s}{l} \sum_{j=0}^{k-1} B_{k,j} \zeta(2s+2l-j, \frac{k+1}{2}),$$

where

$$(8) \quad B_{k,j} = \sum_{m=1}^{k-1} \sum_{p=1}^m \alpha_{k,m} a_p^m P_{m-j}^m \left(m+1-p-\frac{k-1}{2}\right), \quad 0 < j \leq k-1,$$

$$(8') \quad B_{k,0} = \alpha_{k,0} + \sum_{m=1}^{k-1} \sum_{p=1}^m \alpha_{k,m} a_p^m P_m^m (m+1-p - \frac{k-1}{2}).$$

From (8) and (8') and [2] it follows that $B_{k,j}$ are independent on δ_1, δ_2 . In order to obtain $B_{k,j}$ in a simpler form let us consider

$$L(s) = \sum_{n=1}^{\infty} \frac{P_k(n)}{[n + \frac{k-1}{2}]^{2s}}.$$

So we have

$$(k-1)!L(s) = \sum_{j=0}^{k-1} B_{k,j} \zeta(2s-j, \frac{k+1}{2}).$$

From

$$(k-1)! \sum_{n=1}^{\infty} \frac{P_k(n)}{[n + \frac{k-1}{2}]^{2s}} = \sum_{j=0}^{k-1} B_{k,j} \zeta(2s-j, \frac{k+1}{2})$$

we have

$$(k-1)! \sum_{n=1}^{\infty} \frac{P_k(n)}{[n + \frac{k-1}{2}]^{2s}} = \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} B_{k,j} \left(n + \frac{k-1}{2}\right)^j \left(n + \frac{k-1}{2}\right)^{-2s}$$

so that

$$P_k(n) = \sum_{j=0}^{k-1} B_{k,j} \left(n + \frac{k-1}{2}\right)^j$$

which gives, by the binomial theorem,

$$(9) \quad \alpha_{k,m} = \sum_{j=m}^{k-1} B_{k,j} \binom{j}{m} \left(\frac{k-1}{2}\right)^{j-m}, \quad 0 \leq m \leq k-1.$$

From (9), solving a linear system, we get a formula for $B_{k,j}$ by far simpler than (8) and (8'):

$$(10) \quad B_{k,j} = \sum_{p=0}^{k-j-1} (-1)^{k+j+1} \binom{j+p}{j} \left(\frac{k-1}{2}\right)^p (s_{k,j+p+1} + s_{k-1,j+p}),$$

$$0 \leq j \leq k-1.$$

So Theorem 1 is proved. $\square$

5. PROOF OF THEOREM 2

Lemma. *In the above corollary, $B_{k,j} = 0$ if $j = 0$ or if $j = k - 2h$ with $1 \leq h \leq \left[\frac{k}{2}\right]$.*

Proof. First step. From (10) we can immediately see that

$$\begin{aligned} B_{k,0} &= \sum_{p=0}^{k-1} (-1)^{k+1} \left(\frac{k-1}{2} \right)^p ((s_{k-1,p} + s_{k,p+1})) \\ &= \left[\mathcal{R}\left(\frac{1-k}{2}, k-1\right) + \frac{2}{1-k} \mathcal{R}\left(\frac{1-k}{2}, k\right) \right] = 0. \end{aligned}$$

Second step. If we put

$$P_{n,t}(x) = \sum_{i=t}^n \binom{i}{t} s_{n,i} x^i, \quad Q_{n,t}(x) = \sum_{i=1}^n (-1)^{i+1} \frac{n!}{(n-i)!} P_{n-i,t}(x),$$

we have

$$(11) \quad Q_{n,t}(x) + Q_{n,t-1}(x) = \frac{n}{x} P_{n,t}(x).$$

In fact, starting from

$$\begin{aligned} Q_{n,t}(x) + Q_{n,t-1}(x) &= \sum_{i=1}^n (-1)^{i+1} \frac{n!}{(n-i)!} [P_{n-i,t}(x) + P_{n-i,t-1}(x)] \\ &= \sum_{i=1}^n (-1)^{i+1} \frac{n!}{(n-i)!} \sum_{l=t-1}^{n-i} \binom{l+1}{t} s_{n-i,l} x^l \\ &= \sum_{l=t-1}^{n-1} \binom{l+1}{t} x^l \sum_{i=1}^{n-l} (-1)^{i+1} \frac{n!}{(n-i)!} s_{n-i,l} \quad \text{by using (1d)} \\ &= \sum_{l=t-1}^{n-1} \binom{l+1}{t} [s_{n,l} - s_{n+1,l+1}] x^l, \end{aligned}$$

(11) follows using (1a). By using (1a) iteratively in (10) we get

$$B_{k,j} = 2P_{k-1,j}\left(\frac{k-1}{2}\right) - Q_{k-1,j}\left(\frac{k-1}{2}\right)$$

so that, since by (11) with $n = k-1$, $x = \frac{k-1}{2}$, $t = j$

$$(12) \quad 2P_{k-1,j}\left(\frac{k-1}{2}\right) = Q_{k-1,j}\left(\frac{k-1}{2}\right) + Q_{k-1,j-1}\left(\frac{k-1}{2}\right)$$

we have

$$B_{k,j} = Q_{k-1,j-1}\left(\frac{k-1}{2}\right).$$

Third step. Our thesis is equivalent to showing that

$$Q_{k-1,j-1}\left(\frac{k-1}{2}\right) = 0, \quad \text{if } j = k - 2h, \quad h \in \mathbb{N}.$$

By a simple direct computation, $Q_{k-1,k-3} = 0$. By (12) it will be enough to prove that

$$P_{k-1,j}\left(\frac{k-1}{2}\right) = P_{k-1,j-1}\left(\frac{k-1}{2}\right), \quad \text{if } j = k - 2h - 1 > 0, \quad h \geq 1.$$

It is easy to see that

$$P_{n,t}(x) = \frac{x^t}{t!} \frac{d^t}{dx^t} P_{n,0}(x).$$

So our thesis follows, taking $n = k - 1$, $l = h$ if we prove that for $n \in \mathbb{N}$

$$(13) \quad \frac{n}{2(n-2l)} P_{n,0}^{(n-2l)}\left(\frac{n}{2}\right) = P_{n,0}^{(n-2l-1)}\left(\frac{n}{2}\right).$$

Observing that $P_{n,0}(x) = xP_{n-1,0}(x-1)$ we have

$$P_{n,0}^{(n-2l)}\left(\frac{n}{2}\right) = (n-2l)P_{n-1,0}^{(n-2l-1)}\left(\frac{n}{2}-1\right) + \frac{n}{2}P_{n-1,0}^{(n-2l)}\left(\frac{n}{2}-1\right),$$

$$P_{n,0}^{(n-2l-1)}\left(\frac{n}{2}\right) = (n-2l-1)P_{n-1,0}^{(n-2l-2)}\left(\frac{n}{2}-1\right) + \frac{n}{2}P_{n-1,0}^{(n-2l-1)}\left(\frac{n}{2}-1\right),$$

and (13) follows since

$$P_{n-1,0}^{(n-2l)}\left(\frac{n}{2}-1\right) = 0, \quad \forall l \in \mathbb{N}.$$

In fact, $P_{n-1,0}(x) = G(x - (\frac{n}{2} - 1))$ with G a function of parity opposite to n 's one, and so having every derivatives of the same parity of n 's one null. So Lemma is proved. $\square$

Proof of Theorem 2. From the Lemma and Theorem 1 we can easily deduce the following facts:

$Z_k(s)$ is a meromorphic function with a simple pole at $s = \frac{k}{2}$.

If k is even, $Z_k(s)$ has simple poles at most at positive integers $s = \frac{k}{2} - n$, $n \in \mathbb{N}$.

If k is odd, $Z_k(s)$ has simple poles at most at rationals $s = \frac{k}{2} - n$, $n \in \mathbb{N}$.

Furthermore, we can give the following formula for the residues of $Z_k(s)$ at $s_0 = \frac{k}{2} - n_0$, n_0 a nonnegative integer, recalling that the residue of the Hurwitz zeta function at its pole is 1:

$$\frac{1}{(k-1)!} \sum_{h=0}^{\frac{k}{2}-1} \sum_{\substack{l+h=n_0 \\ l \geq 0}} (-1)^l \left(\frac{k-1}{2}\right)^{2l} \binom{n_0 - \frac{k}{2}}{l} B_{k, k-2h-1}.$$

So Theorem 2 is proved. $\square$

Remarks. 1. If k is odd, $Z_k(s)$ vanishes at $s = -n$, $n \in \mathbb{N}$. In fact, from

$$(k-1)!Z_k(-n) = \sum_{l=0}^n (-1)^l \left(\frac{k-1}{2}\right)^{2l} \binom{n}{l} \sum_{j=1}^{k-1} B_{k,j} \zeta(-2n+2l-j, \frac{k+1}{2}),$$

we note that

$$\zeta(-2n+2l-j, \frac{k+1}{2}) = \zeta(-2n+2l-j) - \sum_{t=1}^{\frac{k+1}{2}-1} \frac{1}{t^{-2n+2l-j}}$$

where $\zeta(s)$ is the Riemann zeta function which vanishes at negative even integers, so we have

$$\begin{aligned} \sum_{j=0}^{k-1} B_{k,j} \sum_{l=1}^{\frac{k+1}{2}-1} \frac{1}{t^{-2n+2l-j}} \\ = \sum_{j=1}^{k-1} \sum_{m=1}^{k-1} \sum_{p=1}^m \alpha_{k,m} a_p^m P_{m-j}^m (m+1-p - \frac{k-1}{2}) \sum_{l=1}^{\frac{k+1}{2}-1} \frac{1}{t^{-2n+2l-j}} = 0, \end{aligned}$$

by the same argument used to get (7).

2. From $B_{k,j} = 0$ if $j = k - 2h$ or if $j = 0$ we deduce easily the following relations among the Stirling numbers

$$\sum_{j=k-2h}^{k-1} \binom{j}{k-2h} \left(\frac{k-1}{2}\right)^j (s_{k,j+1} + s_{k-1,j}) = 0, \quad 1 \leq h \leq \frac{k}{2}.$$

Moreover, we also have from (9)

$$(k-1)! = \sum_{j=0}^{k-1} B_{k,j} \left(\frac{k-1}{2}\right)^j.$$

3. The Minakshisundaram-Pleijel zeta function $L_k(s)$ of k -dimensional real projective space $\mathbb{P}^k$ takes the form, in $\Re(s) > \frac{k}{2}$,

$$L_k(s) = \sum_{n=1}^{\infty} \frac{P_k(2n)}{[2n(2n+k-1)]^s}.$$

It is easy to see that

$$4^s (k-1)! L_k(s) = \sum_{l=0}^{\infty} (-1)^l \left(\frac{k-1}{4}\right)^{2l} \binom{-s}{l} \sum_{j=0}^{k-1} 2^j B_{k,j} \zeta(2s+2l-j, \frac{k+3}{4})$$

where $B_{k,j}$ are the same as for $\mathbb{S}^k$.

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