SETS IN THE RANGES OF SUMS FOR PERTURBATIONS OF NONLINEAR m -ACCRETIVE OPERATORS IN BANACH SPACES

ATHANASSIOS G. KARTSATOS

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ABSTRACT. Several results are given involving nonlinear range inclusions of the types $B + D \subset \overline{R(T + C)}$ and $\text{int}(B + D) \subset R(T + C)$, where B, D are subsets of a real Banach space X , the operator $T : X \supset D(T) \rightarrow 2^X$ is at least m -accretive, and the perturbation $C : X \supset D(C) \rightarrow X$ is at least compact, or demicontinuous, or m -accretive. Leray-Schauder degree theory is used in most of the results, and extended versions of recent results of Calvert and Gupta, Morales, Reich, and the author are shown to be possible by using mainly homotopies of compact transformations.

1. INTRODUCTION-PRELIMINARIES

In what follows, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J . An operator $T : X \supset D(T) \rightarrow 2^X$ is called “accretive” if for every $x, y \in D(T)$ there exists $j \in J(x - y)$ such that

$$\langle u - v, j \rangle \geq 0$$

for all $u \in Tx, v \in Ty$. An accretive operator T is “ m -accretive”, if $R(T + \lambda I) = X$ for all $\lambda \in (0, \infty)$. We denote by R and R_+ the sets $(-\infty, \infty)$ and $[0, \infty)$, respectively.

We say that the duality mapping satisfies “Condition (I)” if there exists a function $\phi : X \rightarrow [0, \infty)$ such that $u, v \in X$ implies that

$$\sup\{\|j_1 - j_2\| ; j_1 \in Ju, j_2 \in Jv\} \leq \phi(u - v).$$

This concept was introduced by Calvert and Gupta in [3, Definition 1.1]. They actually showed that the duality mapping in the space $L^p(\Omega)$, $p \in (1, \infty)$, satisfies Condition (I), where Ω is a bounded domain in R^n .

Morales gave three results in [10, Theorems 1–3] involving Banach spaces whose duality mappings satisfy Condition (I). These results extend various results of the author [5] and several of the references therein. The author considered in [5] Banach spaces X with uniformly convex dual spaces. Since, even

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in that case, it is not known whether the duality mapping J actually satisfies a global condition like (I), Morales's results are, in a sense, independent of ours in [5]. Condition (I) was also used by Torrejón in [13].

One of our purposes here (Theorem 1) is to extend Theorems 1 and 2 of Morales [10] by eliminating the assumption that the duality mapping J satisfy Condition (I). We do this by a simple modification in the hypotheses of Morales's statements. Furthermore, this result provides conditions such that $B + D \subset R(T + C)$ and $\text{int}(B + D) \subset R(T + C)$ for two otherwise arbitrary sets $B, D \subset X$. The operator T in all these results is m -accretive with compact (completely continuous) resolvents and C is continuous (demicontinuous) and bounded. Proposition 1 is a statement that parallels Proposition 1.10 in Reich's paper [11] concerning ranges of sums of accretive operators. We consider duality mappings satisfying Condition (I). In this connection, the reader is also referred to Torrejón's paper [13, Theorem 2.5] and Theorem 3 in Morales's paper [10]. Finally, we give a new result (considering jointly Theorems 2 and 3) involving ranges of sums in uniformly convex Banach spaces. These sums involve compact perturbations of m -accretive operators. For a survey article about compactness and accretiveness, we cite the paper [7].

We denote by $B_r(0)$ the open ball with center at zero and radius $r > 0$. For an m -accretive operator T , the "resolvents" $J_\lambda : X \rightarrow D(T)$ of T are defined by $J_\lambda = (I + \lambda T)^{-1}$ for all $\lambda \in (0, \infty)$. The "Yosida approximants" $T_\lambda : X \rightarrow X$ of T are defined by $T_\lambda = \frac{1}{\lambda}(I - J_\lambda)$. For $x \in X$, we define $|Tx|$ by

$$|Tx| = \lim_{\lambda \rightarrow 0} \|T_\lambda x\|.$$

Some of the main properties of J_λ and T_λ are given below:

1. $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$ for all $x, y \in X$.
2. $\|J_\lambda x - x\| = \lambda \|T_\lambda x\| \leq \lambda \inf\{\|y\| : y \in Tx\}$ for all $x \in D(T)$.
3. T_λ is m -accretive on X and $\|T_\lambda x - T_\lambda y\| \leq \frac{2}{\lambda} \|x - y\|$ for all $\lambda > 0$, $x, y \in X$.
4. $T_\lambda x \in TJ_\lambda x$ for all $x \in X$.
5. $\|T_\lambda x\| \leq |Tx|$ for all $x \in D(T)$.

For these facts the reader is referred to Barbu [1] and Lakshmikantham and Leela [8].

In the sequel, "continuous" means "strongly continuous" and the symbol " $\rightarrow$ " (" $\rightharpoonup$ ") means strong (weak) convergence. The symbols ∂D , $\overline{D}$, and $\text{int } D$ denote the strong boundary, closure, and interior of the set D , respectively. An accretive operator T is called "strongly accretive" if there exists a constant $\alpha > 0$ such that: for each $x, y \in D(T)$ there exists $j \in J(x - y)$ such that

$$\langle u - v, j \rangle \geq \alpha \|x - y\|^2$$

for all $u \in Tx$, $v \in Ty$. An operator $T : X \supset D(T) \rightarrow X$ is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. It is called "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on $D(T)$.

Definition 1. Let $U : [0, 1] \times G \rightarrow X$, where G is a bounded subset of X . We say that the mapping $U(t, x)$ is G -continuous at t_0 if for every $\epsilon > 0$ there

exists $\delta(\epsilon, t_0) > 0$ with the property: $\|U(t, x) - U(t_0, x)\| < \epsilon$ for every $t \in [0, 1]$ with $|t - t_0| < \delta(\epsilon, t_0)$ and every $x \in G$. We say that U is G -continuous on $[0, 1]$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with the property: $\|U(t_1, x) - U(t_2, x)\| < \epsilon$ for every $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta(\epsilon)$ and every $x \in G$.

We now state a fundamental lemma that is an easy consequence of the Leray-Schauder theory. It can be found in Lloyd [9, Theorem 4.4.11].

Lemma 1. *Let X be a Banach space, and consider the equation*

$$(1.1) \quad u - U(t, u) = 0,$$

for which we assume the following:

(i) $U : [0, 1] \times G \rightarrow X$ is compact in its second variable for each $t \in [0, 1]$, where G is a closed, convex, and bounded subset of X containing the origin in its interior. Furthermore, $U(t, x)$ is G -continuous on $[0, 1]$.

(ii) $U(0, \partial G) \subset G$.

(iii) $U(t, x) \neq x$, $t \in [0, 1]$, $x \in \partial G$.

Then (1.1) has a solution in G for $t = 1$.

The following lemma follows easily from Browder's Lemma 2 in [2]. Versions of it have also been used by the author [5], Reich [11], and Morales [10].

Lemma 2. *Let $\{x_n\} \subset X$ and $\{\alpha_n\}$ be a sequence of positive numbers tending to zero as $n \rightarrow \infty$. Fix $r > 0$, and assume that for every $h \in X$ with $\|h\| \leq r$ there exists a constant C_h such that*

$$\langle h, j \rangle \leq \alpha_n \|x_n\| + C_h$$

for all n and some $j = j_n \in Jx_n$. Then the sequence $\{x_n\}$ is bounded.

In the following lemma we show that a seemingly stronger, but useful, result holds in spaces where J satisfies Condition (I).

Lemma 3. *Assume that X^* is uniformly convex, or that the duality mapping J satisfies Condition (I). Let $\{x_n\} \subset X$ and $\{\alpha_n\}$ be a sequence of positive numbers tending to zero as $n \rightarrow \infty$. Fix $r > 0$, and assume that for every $h \in X$ with $\|h\| \leq r$ there exists a constant C_h and $u_h \in X$ such that*

$$\langle h, j \rangle \leq \alpha_n \|x_n\| + C_h$$

for all n and some $j = j_{n,h} \in J(x_n - u_h)$. Then the sequence $\{x_n\}$ is bounded.

Proof. Let J satisfy Condition (I), and let $j_n \in Jx_n$ be given. Let $h \in X$ be such that $\|h\| \leq r$. Then

$$\|j_{n,h} - j_n\| \leq \phi(u_h)$$

and

$$\begin{aligned} \langle h, j_{n,h} \rangle &= \langle h, j_{n,h} - j_n \rangle + \langle h, j_n \rangle \\ &\leq \alpha_n \|x_n\| + C_h, \end{aligned}$$

implying

$$\begin{aligned} \langle h, j_n \rangle &\leq \alpha_n \|x_n\| + [C_h + \|h\|\phi(u_h)] \\ &\leq \alpha_n \|x_n\| + [C_h + r\phi(u_h)], \end{aligned}$$

which is the condition in Lemma 2 with another constant C_h there. If X^* is uniformly convex, the proof follows easily from the proof of Proposition (1) of Browder [2].

We denote by Γ the set of all functions $\beta : R_+ \rightarrow R_+$ such that $\beta(r) \rightarrow 0$ as $r \rightarrow \infty$.

2. MAIN RESULTS

Our first theorem is a new result on ranges of sums. It extends a result of the author [5, Theorem 1] and two results of Morales [10, Theorems 1 and 2]. Note that X is a general Banach space whose duality mapping J does not have to satisfy Condition (I).

Theorem 1. *Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : X \supset D(C) \rightarrow X$ bounded with $D(C) \supset D(T)$. Let $\beta \in \Gamma$, $B \subset X$, $D \subset X$, $Q > 0$ be such that:*

(i) *for every $(p, q) \in B \times D$ there exists a constant $K(p, q) > 0$ such that*

$$(2.1) \quad \begin{aligned} \langle u - p, j \rangle &\geq -K(p, q) - \beta(\|x\|)\|x\|, \\ \langle Cx - q, j \rangle &\geq -K(p, q) - \beta(\|x\|)\|x\| \end{aligned}$$

for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in Jx$.

Assume further one of the following statements:

- (a) *$(I + T)^{-1}$ is compact and C is continuous with $D(C) = \overline{D(T)}$.*
- (b) *X^* is uniformly convex, $(I + T)^{-1}$ is completely continuous, $D(C) = X$, and C is demicontinuous.*

Then $B + D \subset \overline{R(T + C)}$ and $\text{int}(B + D) \subset R(T + C)$.

Proof. Assume (a), let $s \in X$, and consider the approximating problem

$$(2.2) \quad Tx + Cx + \frac{1}{n}x \ni s$$

or, equivalently, the problem

$$(2.3) \quad u + C(T + \frac{1}{n}I)^{-1}u - s = 0.$$

This equation leads to the homotopy equation

$$(2.4) \quad u + t(C(T + \frac{1}{n}I)^{-1}u - s) = 0$$

for $(t, u) \in [0, 1] \times X$. Since the mapping $U(t, u) \equiv -t(C(T + \frac{1}{n}I)^{-1}u - s)$ satisfies (i), (ii) of Lemma 1 on every ball $G = \overline{B_\mu(0)}$, $\mu > 0$, the equation $u - U(1, u) = 0$ (hence inclusion (2.2)) will be solvable, by Lemma 1, if all possible solutions of (2.4) are uniformly bounded w.r.t. $t \in (0, 1]$. To show this, let $\{u_m\}_{m=1}^\infty$ be such that for each m the point u_m solves (2.4) for $t = t_m \in (0, 1]$, and assume that $\{u_m\}_{m=1}^\infty$ is unbounded. Then $x_m \equiv (T + \frac{1}{n}I)^{-1}u_m$ solves

$$(2.5) \quad v_m + t_m Cx_m + \frac{1}{n}x_m = t_ms,$$

where v_m is some point in Tx_m . Also, since

$$\|u_m\| = \|v_m + \frac{1}{n}x_m\| \leq \|Cx_m\| + \|s\|,$$

we have that if $\{x_m\}_{m=1}^\infty$ is bounded, so is $\{u_m\}_{m=1}^\infty$, by the boundedness of C , i.e., a contradiction. Thus, $\{x_m\}_{m=1}^\infty$ is unbounded and we let m be such that $\|x_m\| \geq Q$. Let $(p, q) \in B \times D$, $s = p + q$, and note that (2.1) holds for $x = x_m$. Then we have, for an appropriate $j \in Jx_m$,

$$\begin{aligned} \frac{1}{n}\|x_m\|^2 &= -\langle v_m - t_m p, j \rangle - t_m \langle Cx_m - q, j \rangle \\ &\leq -\langle v_m - p, j \rangle - (1 - t_m) \langle p, j \rangle - t_m \langle Cx_m - q, j \rangle \\ &\leq 2K(p, q) + 2\beta(\|x_m\|)\|x_m\| + \|p\|\|x_m\|. \end{aligned}$$

This shows the boundedness of $\{x_m\}_{m=1}^\infty$ and implies the solvability of (2.2). It is easy to see now that if $\{x_n\}$ solves (2.2) with $\|x_n\| \geq Q$ and $s \in B + D$, then

$$\frac{1}{n}\|x_n\|^2 \leq 2K(p, q) + 2\beta(\|x_n\|)\|x_n\|$$

which shows that $\frac{1}{n}x_n \rightarrow 0$ as $n \rightarrow \infty$ and implies that $s \in \overline{R(T + C)}$ or $B + D \subset \overline{R(T + C)}$.

To prove our second conclusion, let $s \in \text{int}(B + D)$ and let $r > 0$ be such that $s + h = p_h + q_h \in B + D$ for all $h \in \overline{B_r(0)}$, where $(p_h, q_h) \in B \times D$. Fix such an h . Then if x_n solves (2.2) with $\|x_n\| \geq Q$, we have, for an appropriate $j = j_n \in Jx_n$,

$$\langle u_n - p_h, j \rangle + \langle Cx_n - q_h, j \rangle + \frac{1}{n}\langle x_n, j \rangle = -\langle h, j \rangle,$$

for some $u_n \in Tx_n$, which implies

$$\langle h, j \rangle \leq 2K(p_h, q_h) + 2\beta(\|x_n\|)\|x_n\|.$$

Letting $\alpha_n = 2\beta(\|x_n\|)$ and $C_h = 2K(p_h, q_h)$, we obtain from Lemma 2 that $\{x_n\}$ is bounded. Since

$$(2.6) \quad x_n = (T + I)^{-1} \left[-Cx_n + s + \left(\frac{n-1}{n} \right) x_n \right],$$

we have that $\{x_n\}$ lies inside a compact set. As such, it contains a subsequence $\{x_{n_k}\}$ converging to some $x_0 \in \overline{D(T)}$ as $k \rightarrow \infty$. Since C is continuous, $D(C) \supset \overline{D(T)}$, and T is closed, we have that $s \in Tx_0 + Cx_0$. Thus, $\text{int}(B + D) \subset \overline{R(T + C)}$.

Now, assume (b) and consider inclusion (2.2) rewritten as follows:

$$(2.7) \quad x - (T + \frac{1}{n}I)^{-1}(s - Cx) = 0.$$

This leads to the homotopy equation

$$(2.8) \quad x - (T + \frac{1}{n}I)^{-1}(t(s - Cx)) = 0.$$

Since X is reflexive, the mapping $U(t, x) \equiv (T + \frac{1}{n}I)^{-1}(t(s - Cx))$ satisfies (i), (ii) of Lemma 1 in every large ball around zero. In fact, if we let $y_0 =$

$(T + \frac{1}{n}I)^{-1}(0)$, and then take $\mu \in (\|y_0\|, \infty)$, the conditions (i), (ii) of Lemma 1 are satisfied on any $G = \overline{B_\mu(0)}$. Indeed, $U(0, \partial B_\mu(0)) = \{y_0\}$ and $\|y_0\| < \mu$ implies that $U(0, \partial B_\mu(0)) \subset B_\mu(0)$. Since C is bounded, there exists a constant $L > 0$ such that $\|Cx\| \leq L$ for $x \in \overline{B_\mu(0)}$. Then, for such x and $y_{t_i}(x) \equiv (T + \frac{1}{n}I)^{-1}(t_i(s - Cx)) = (nT + I)^{-1}(t_in(s - Cx))$, $i = 1, 2$, we have

$$\begin{aligned} \|y_{t_1}(x) - y_{t_2}(x)\| &\leq \|t_1(ns - Cx) - t_2(ns - Cx)\| \\ &\leq |t_1 - t_2|(n\|s\| + \|Cx\|) \leq |t_1 - t_2|(n\|s\| + L), \end{aligned}$$

proving the G-continuity of $U(t, x)$ on $[0, 1]$. Thus, in order to apply Lemma 1 on any large ball as above, we must show that the solutions of the homotopy equation (2.8) are uniformly bounded on $[0, 1]$. To this end, let $\{x_m\}_{m=1}^\infty$ be an unbounded sequence of solutions of (2.8), where $t = t_m \in (0, 1]$. Let $\|x_m\| \geq Q$ for some m . Then we have

$$v_m + t_m Cx_m + \frac{1}{n}x_m = t_ms,$$

i.e., equation (2.5). Thus the proof of the uniform boundedness of the solutions of (2.8) follows exactly as above, and it is therefore omitted. Letting now x_n solve (2.2) with $\|x_n\| \geq Q$ and $s \in B + D$, we obtain again that $\frac{1}{n}x_n \rightarrow 0$ as $n \rightarrow \infty$, which implies that $s \in \overline{R(T + C)}$. The rest of the proof follows exactly as above because the argument involving (2.6) is almost identical to that of case (a). In fact, the only difference is that we now need the demiclosedness of the operator T , which is guaranteed by the uniform convexity of the dual space X^* (cf. Barbu [1, Proposition 3.5]). This completes the proof.

We note here that the solvability of the inclusion (2.2), in the proof of Theorem 1, does not require the full strength of the conditions (2.1). In fact, it suffices to assume instead that there exist constants $c > 0$, $d > 0$ such that $\langle u, j \rangle \geq -c\|x\| - d$ and $\langle Cx, j \rangle \geq -c\|x\| - d$ for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in Jx$. The proof under this assumption goes through easily in both cases (a) and (b). In this connection, the reader is referred to Lemma 2.1 of Calvert and Gupta [3].

The following corollary uses $B = R(T)$ and $D = R(C)$ in Theorem 1. We use the symbol $A \simeq B$ to denote the fact that $\overline{A} = \overline{B}$ and $\text{int } A = \text{int } B$.

Corollary 1. *Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : X \supset D(C) \rightarrow X$ bounded with $D(C) \supset D(T)$. Let $\beta \in \Gamma$, $Q > 0$ be such that:*

- (i) *for every $x_0 \in D(T)$, $y_0 \in D(C)$, and $u_0 \in Tx_0$ there exists a constant $K(u_0, y_0) > 0$ such that*

$$\begin{aligned} \langle u - u_0, j \rangle &\geq -K(u_0, y_0) - \beta(\|x\|)\|x\|, \\ \langle Cx - Cy_0, j \rangle &\geq -K(u_0, y_0) - \beta(\|x\|)\|x\| \end{aligned}$$

for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in Jx$.

Assume further (a) or (b) of Theorem 1. Then $R(T + C) \simeq R(T) + R(C)$.

Proof. Take $B = R(T)$ and $D = R(C)$ in the proof of Theorem 1. Then $R(T) + R(C) \subset \overline{R(T + C)}$ and $\text{int}(R(T) + R(C)) \subset R(T + C)$. This proves our assertion.

In the following corollaries one of the sets B , D is equal to $\{0\}$.

Corollary 2. Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : X \supset D(C) \rightarrow X$ bounded with $D(C) \supset D(T)$. Let $\beta \in \Gamma$, $C_1 > 0$, $Q > 0$ be such that:

- (i) for every $x_0 \in D(T)$ and every $u_0 \in Tx_0$ there exists a constant $K(u_0) > 0$ such that

$$\langle u - u_0, j \rangle \geq -K(u_0) - \beta(\|x\|)\|x\|,$$

$$\langle Cx, j \rangle \geq -C_1 - \beta(\|x\|)\|x\|$$

for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in Jx$.

Assume further (a) or (b) of Theorem 1. Then $R(T) \subset \overline{R(T+C)}$ and $\text{int } R(T) \subset R(T+C)$.

Proof. Just take $B = R(T)$ and $D = \{0\}$ in the proof of Theorem 1.

Corollary 3. Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : X \supset D(C) \rightarrow X$ bounded with $D(C) \supset D(T)$. Let $\beta \in \Gamma$, $C_1 > 0$, $Q > 0$ be such that:

- (i) for every $y_0 \in D(C)$ there exists a constant $K(y_0) > 0$ such that

$$\langle u, j \rangle \geq -C_1 - \beta(\|x\|)\|x\|,$$

$$\langle Cx - Cy_0, j \rangle \geq -K(y_0) - \beta(\|x\|)\|x\|$$

for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in Jx$.

Assume further (a) or (b) of Theorem 1. Then $R(C) \subset \overline{R(T+C)}$ and $\text{int } R(C) \subset R(T+C)$.

Proof. Just take $B = \{0\}$ and $D = R(C)$ in Theorem 1.

It is important to note here that there are several other corollaries to Theorem 1. For example, one could take $B = R(C)$ and $D = R(T)$ in the proof of Theorem 1 with u_0 , Cy_0 interchanged in the left-hand sides of the two operator inequalities of Corollary 1. This would not affect the conclusion of Corollary 1. Similar remarks apply to Corollaries 2 and 3.

Certain improvements of Theorem 1 are possible. For example, let us consider the following condition instead of condition (i) of Theorem 1:

- (j) for every $(p, q) \in B \times D$ there exists $a = a(p, q) \in X$ and a constant $K(a, p, q) > 0$ such that

$$(2.9) \quad \langle u - p, j \rangle \geq -K(a, p, q) - \beta(\|x\|)\|x\|,$$

$$\langle Cx - q, j \rangle \geq -K(a, p, q) - \beta(\|x\|)\|x\|$$

for all $x \in D(T)$ with $\|x\| \geq Q$, all $u \in Tx$, and some $j \in J(x - a)$.

It can be easily shown, as in the proof of Theorem 1, that the first conclusion of Theorem 1 is true under condition (j). The second conclusion of Theorem 1 is also true under (j) if we assume, in addition, that J satisfies Condition (I). To see this, let $s \in \text{int}(B + D)$, and let $s + h = p_h + q_h \in B + D$, with $p_h \in B$, $q_h \in D$, and $h \in \overline{B_r(0)}$. Let x_n solve (2.2) with $\|x_n\| \geq Q$ and note that (2.9) holds with $p = p_h$, $q = q_h$, and $a = a(h)$. Then, for some $j \in J(x_n - a(h))$,

$$\langle h, j \rangle \leq \frac{1}{4}\|a(h)\|^2 + 2K(a(h), p_h, q_h) + 2\beta(\|x_n\|)\|x_n\|,$$

where we have used the fact that $\langle x_n, j \rangle \geq -\frac{1}{4}\|a(h)\|^2$. Letting $\alpha_n = 2\beta(\|x_n\|)$ and $C_h = \frac{1}{4}\|a(h)\|^2 + 2K(a(h), p_h, q_h)$, we obtain the boundedness of $\{x_n\}$

from Lemma 3. The rest of the proof follows exactly as the corresponding part of the proof of Theorem 1. It is therefore omitted. Naturally, one can now state 3 corollaries as in the case of Theorem 1. This remark provides an extension of Theorem 1 of the author [5] and contains, as special cases, Theorems 1 and 2 of Morales [10].

Proposition 1 below is a general result concerning the existence of sets in the ranges of sums of accretive operators. It is like Proposition 1.10 in Reich's paper [11], where X^* is assumed to be uniformly convex. It also extends Theorem 1.2 of Calvert and Gupta [3], Theorem 3 of Morales [10], and Theorem 2.5 of Torrejón [13].

Proposition 1. *Assume that every non-empty, closed, convex, and bounded subset of X has the fixed point property with respect to non-expansive self-mappings. Let J satisfy Condition (I). Let $T : X \supset D(T) \rightarrow 2^X$ be accretive and $C : X \supset D(C) \rightarrow 2^X$ accretive and such that $D(C) \supset D(T)$ and $T+C$ is m -accretive. Let $\beta \in \Gamma$, $B \subset X$, $D \subset X$, $Q > 0$ be such that:*

- (i) *for each $(p, q) \in B \times D$ there exist $a, b \in X$ and $K_1(a, p) > 0$, $K_2(b, q) > 0$ such that*

$$(2.10) \quad \begin{aligned} \langle u - p, j_1 \rangle &\geq -K_1(a, p) - \beta(\|x\|)\|x\|, \\ \langle Cx - q, j_2 \rangle &\geq -K_2(b, q) - \beta(\|x\|)\|x\| \end{aligned}$$

for all $x \in D(T+C)$, with $\|x\| \geq Q$, all $u \in Tx$, and some $j_1 \in J(x-a)$, $j_2 \in J(x-b)$.

Assume that for each $f \in B+D$ the sequence $\{u_n\}$ is bounded, where $u_n \in Tx_n$, $v_n \in Cx_n$, and x_n satisfies

$$(2.11) \quad u_n + v_n + \frac{1}{n}x_n = f.$$

Then $B+D \subset \overline{R(T+C)}$ and $\text{int}(B+D) \subset R(T+C)$.

Proof. Since $T+C$ is m -accretive, (2.11) has a solution x_n for all $f \in B+D$. Assume that $\{u_n\}$ is bounded, and let $f = p+q$ with $p \in B$ and $q \in D$. Let (2.10) hold with x_n instead of x , and let $j \in Jx_n$. Then

$$(2.12) \quad \begin{aligned} \langle u_n - p, j \rangle &= \langle u_n - p, j_1 \rangle + \langle u_n - p, j - j_1 \rangle \\ &\geq -K_1(a, p) - \beta(\|x_n\|)\|x_n\| - \|u_n - p\|\phi(\|a\|). \end{aligned}$$

Similarly,

$$(2.13) \quad \langle v_n - q, j \rangle \geq -K_2(b, q) - \beta(\|x_n\|)\|x_n\| - \|v_n - q\|\phi(\|b\|).$$

Adding (2.12) and (2.13), we obtain

$$(2.14) \quad \begin{aligned} \langle \frac{1}{n}x_n, j \rangle &\leq K_1(a, p) + K_2(b, q) + 2\beta(\|x_n\|)\|x_n\| \\ &\quad + \|u_n - p\|\phi(\|a\|) + \|v_n - q\|\phi(\|b\|). \end{aligned}$$

Since $\{u_n\}$ is bounded and $\|v_n\| \leq \|u_n\| + \frac{1}{n}\|x_n\| + \|f\|$, we see that there exist constants C_1, C_2 such that

$$(2.15) \quad \frac{1}{n}\|x_n\|^2 \leq C_1 + \frac{1}{n}C_2\|x_n\| + 2\beta(\|x_n\|)\|x_n\|.$$

This implies that $\frac{1}{n}\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$ and $B+D \subset \overline{R(T+C)}$.

To show our second conclusion, let $f \in \text{int}(B + D)$ and $f + h = p_h + q_h$ with $p_h \in B$, $q_h \in D$ for all $h \in \overline{B_r(0)}$. Then, as above,

$$-\langle \frac{1}{n}x_n + h, j \rangle \geq -C_1(h) - \frac{1}{n}C_2(h)\|x_n\| - 2\beta(\|x_n\|)\|x_n\|,$$

for some $j \in Jx_n$, which, by the boundedness of $\{\frac{1}{n}x_n\}$ ($\frac{1}{n}x_n \rightarrow 0$ as $n \rightarrow \infty$), gives

$$\langle h, j \rangle \leq \overline{C}_1(h) + 2\beta(\|x_n\|)\|x_n\|,$$

where $\overline{C}_1(h)$ is a new constant. By Lemma 2, $\{x_n\}$ is bounded. By Lemma 1 of Reich and Torrjon [12], $f \in B + D$. This ends the proof.

The reader is referred to the paper of Calvert and the author [4], and the references therein, for conditions concerning the m-accretiveness of the sum of two m-accretive operators. The following corollary contains a condition that ensures the boundedness of $\{u_n\}$ in the statement of Proposition 1. This corollary and Corollary 5 are related to Theorem 1.2 of Calvert and Gupta [3].

Corollary 4. *Assume that every non-empty, closed, convex, and bounded subset of X has the fixed point property with respect to non-expansive self-mappings. Let J satisfy Condition (I). Let $T : X \supset D(T) \rightarrow 2^X$ be accretive and $C : X \supset D(C) \rightarrow 2^X$ accretive and such that $D(C) \supset D(T)$ and $T + C$ is m-accretive. Let (i) of Proposition 1 hold. Assume further that there exist constants $K \in (0, 1)$, $C_1 > 0$, $C_2 > 0$ such that*

$$(2.16) \quad \langle y, j \rangle \geq -K\|z\|^2 - C_1\|z\| - C_2$$

for every $x \in D(T + C)$, $y \in Ax$, and $z \in Cx$ and some $j \in Jz$.

Then $B + D \subset \overline{R(T + C)}$ and $\text{int}(B + D) \subset R(T + C)$.

Proof. In view of Proposition 1, it suffices to show the boundedness of $\{u_n\}$ in (2.11) with $f \in B + D$. To this end, we evaluate an appropriate functional $j \in Jv_n$ on the sides of (2.11) to obtain

$$\langle u_n, j \rangle + \|v_n\|^2 + \frac{1}{n}\langle x_n, j \rangle = \langle f, j \rangle$$

and, using condition (2.16),

$$\|v_n\|^2 \leq K\|v_n\|^2 + C_1\|v_n\| + \frac{1}{n}\|x_n\|\|v_n\| + \|f\|\|v_n\| + C_2,$$

or, assuming that $\|v_n\| \geq 1$,

$$(2.17) \quad \|v_n\| \leq C_3 + \left(\frac{C_4}{n}\right)\|x_n\|$$

and

$$\begin{aligned} \|u_n\| &= \left\| f - \frac{1}{n}x_n - v_n \right\| \\ (2.18) \quad &\leq \|f\| + \frac{1}{n}\|x_n\| + C_3 + \left(\frac{C_4}{n}\right)\|x_n\| \\ &= C_5 + \left(\frac{C_6}{n}\right)\|x_n\|. \end{aligned}$$

Combining this with (2.14), we can see now that (2.15) is true. This shows the boundedness of $\{\frac{1}{n}x_n\}$ and, in turn, the boundedness of $\{u_n\}$ and $\{v_n\}$. The proof is complete.

Corollary 5. Assume that every non-empty, closed, convex, and bounded subset of X has the fixed point property with respect to non-expansive self-mappings. Let J satisfy Condition (I). Let $T : X \supset D(T) \rightarrow 2^X$ be accretive and $C : X \supset D(C) \rightarrow 2^X$ accretive and such that $D(C) \supset D(T)$ and $T + C$ is m -accretive. Assume that for each $f \in R(T) + R(C)$ the sequence $\{u_n\}$ is bounded, where $u_n \in Tx_n$, $v_n \in Cx_n$, and x_n satisfies

$$u_n + v_n + \frac{1}{n}x_n = f.$$

Then $R(T + C) \simeq R(T) + R(C)$.

Proof. It suffices to take $B = R(T)$, $D = R(C)$ in Proposition 1. Then Assumption (i) there is trivially satisfied, by the accretiveness of the operators T , C , and we conclude that $R(T) + R(C) \subset \overline{R(T + C)}$ and $\text{int}(R(T) + R(C)) \subset R(T + C)$. Since $R(T + C) \subset R(T) + R(C)$, the proof is complete.

At this point we would like to point out the main difference between the theory of zeros of perturbations of accretive operators and the theory of ranges of sums of such operators. In the theory of zeros, one shows that a solution $\{x_n\}$ of an equation of the type $Tx_n + Cx_n + \frac{1}{n}x_n = p$ (or its multi-valued equivalent) satisfies $\lim_{n \rightarrow \infty} \frac{x_n}{n} = 0$. This implies that $p \in \overline{R(T + C)}$. In the theory of ranges of sums, one solves first the approximate equation above for $p \in \text{int} B$ and then shows that $p \in R(T + C)$ by using additional conditions ensuring the boundedness of the sequence $\{x_n\}$ and the fact that $p \in Tx_0 + Cx_0$ with $x_n \rightarrow x_0$. As we have already seen above, the boundedness of the sequence $\{x_n\}$ is achieved via use of the Uniform Boundedness Principle. The following result is an attempt to make these differences, or similarities, more transparent by looking at the problem $\text{int} B \subset R(T + C)$ from a more “localized” point of view. It will then become obvious that the theory of ranges of sums is nothing else than a part of the theory of zeros of operators with “point conditions” for $p \in R(T + C)$ replaced by “set conditions” that ensure that $p \in R(T + C)$ for all $p \in$ (some set) B . Possible improvements in the assumptions of our previous results will be suggested by these considerations. The symbol $\text{co} A$ denotes the convex hull of the set A .

Theorem 2. Assume that X is uniformly convex. Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : \overline{D(T)} \rightarrow X$ completely continuous. Let $\beta \in \Gamma$, $B \subset X$, and $D \subset X$, and assume the following condition:

- (i) For each $s \in \text{int}(B + D)$ there exists $r(s) > 0$, with $s + \overline{B_{r(s)}(0)} \subset B + D$, $Q(s) > 0$ with the property: $s + h = p_h + q_h$, $p_h \in B$, $q_h \in D$, for some $h \in \overline{B_{r(s)}(0)}$, implies the existence of $K(s + h) > 0$ such that

$$(2.19) \quad \begin{aligned} \langle u - p_h, j \rangle &\geq -K(s + h) - \beta(\|x\|)\|x\|, \\ \langle Cx - q_h, j \rangle &\geq -K(s + h) - \beta(\|x\|)\|x\| \end{aligned}$$

for all $x \in D(T)$ with $\|x\| \geq Q(s)$, all $u \in Tx$, and some $j \in Jx$.

Then $\text{int}(B + D) \subset R(T + C)$.

Theorem 3, which follows, complements Theorem 2 above. It is given separately because it uses weaker assumptions on the space and the operator C . Its proof is omitted because it closely resembles the corresponding part of the proof of Theorem 1.

Theorem 3. Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : \overline{D(T)} \rightarrow X$ compact. Let $\beta \in \Gamma$, $B \subset X$, and $D \subset X$ and assume the following condition:

- (i) for each $s = p + q \in B + D$, with $p \in B$, $q \in D$, there exists $Q(s) > 0$ and $K(s) > 0$ such that

$$(2.20) \quad \begin{aligned} \langle u - p, j \rangle &\geq -K(s) - \beta(\|x\|)\|x\|, \\ \langle Cx - q, j \rangle &\geq -K(s) - \beta(\|x\|)\|x\| \end{aligned}$$

for all $x \in D(T)$ with $\|x\| \geq Q(s)$, all $u \in Tx$, and some $j \in Jx$.

Then $B + D \subset \overline{R(T + C)}$.

We should notice here that $Q(s)$ in (i) of Theorem 2 depends only on s while h is lying in a small neighborhood $\overline{B_{r(s)}(0)}$. Naturally, $s + \overline{B_r(0)} \subset B + D$, for all small $r > 0$, whenever $s \in \text{int}(B + D)$. Also, B could be just a singleton and $D = \{0\}$ in (i) of Theorem 3.

Proof of Theorem 2. We first solve the inclusion (2.2) for $s \in \text{int}(B + D)$. This can be done exactly as in Theorem 1 because the operator

$$H(t, u) \equiv u + t(C(T + \frac{1}{n}I)^{-1}u - s)$$

is again a homotopy of compact transformations. We let $\{x_n\}$ denote a solution of (2.2). Then $\{x_n\}$ is bounded by Lemma 2, as in the proof of Theorem 1, by letting $h \in \overline{B_{r(s)}(0)}$. Since X is reflexive, there exists $x_0 \in \overline{\text{co } D(T)} = \overline{D(T)}$ such that $x_n \rightharpoonup x_0$. The fact that $x_0 \in D(T)$ and $s \in Tx_0 + Cx_0$ follows from the following lemma, which is an easy multivalued extension of Lemma 1 of the author [6].

Lemma 4. Let X be uniformly convex. Let $T : X \supset D(T) \rightarrow 2^X$, $C : \overline{D(T)} \rightarrow X$ be m -accretive and completely continuous, respectively. Assume that there exists a sequence $\{\alpha_n\}$ of positive constants and a sequence $\{x_n\} \subset \overline{B_b(0)} \cap D(T)$ such that $\alpha_n \rightarrow 0$ and $x_n \rightarrow x_0$ as $n \rightarrow \infty$ and

$$u_n + Cx_n + \alpha_n x_n = s$$

for some sequence $u_n \in Tx_n$, where s is a point in X . Then $x_0 \in D(T)$ and $u_n \rightarrow u_0 = -Cx_0 + s$, where $u_0 \in Tx_0$.

3. DISCUSSION

It is possible to obtain results of the type $\text{int } B \subset R(T + C)$ by solving an equation of the type

$$(2.21) \quad Tx + Cx + \frac{1}{n}x \ni p + h,$$

where $p \in \text{int } B$ and h is sufficiently small. To show the convergence of a sequence $\{x_n\}$ solving (2.21), we may assume strong accretiveness assumptions on T , or other appropriate coercivity conditions. This method does not involve the application of the Uniform Boundedness Principle. Nevertheless, one would have to assume conditions on the operators T , C that involve an entire small neighborhood of the point p as in Theorem 2.

Applications of the results in this paper include nonlinear elliptic boundary value problems in L^p spaces ($p \in (0, \infty)$). Such a problem was studied by

Calvert and Gupta [3]. In that paper an m -accretive operator A_p with compact resolvents is constructed in such spaces. Also, sufficient conditions are given for a function $f \in L^p(\Omega)$ to belong to the set $\text{int } R(A_p)$. Here Ω is an open bounded domain of R^n with smooth boundary.

It would be quite interesting to know whether the methods of ranges of sums can actually be used to obtain further improvements of the results concerning the existence of solutions of nonlinear operator equations involving compact perturbations and compact resolvents of accretive operators.

Naturally, extensions of the present results are possible if one replaces the various compact mappings by appropriate condensing mappings and uses the generalized degree for such mappings instead of the Leray-Schauder degree.

It is also natural to expect that the results of this paper can be extended to suitable combinations involving A -proper mappings by using appropriate finite-dimensional arguments, via methods of approximation solvability, if the underlying Banach spaces possess nice projection schemes.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620-5700
E-mail address: hermes@gauss.math.usf.edu