TRIANGULAR UHF ALGEBRAS OVER ARBITRARY FIELDS

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ABSTRACT. Let K be an arbitrary field. Let (q_n) be a sequence of positive integers, and let there be given a family $\{\Psi_{nm} | n \geq m\}$ of unital K -monomorphisms $\Psi_{nm}: T_{q_m}(K) \rightarrow T_{q_n}(K)$ such that $\Psi_{np}\Psi_{pm} = \Psi_{nm}$ whenever $m \leq n$, where $T_{q_n}(K)$ is the K -algebra of all $q_n \times q_n$ upper triangular matrices over K . A *triangular UHF (TUHF) K -algebra* is any K -algebra that is K -isomorphic to an algebraic inductive limit of the form $\mathcal{S} = \varinjlim (T_{q_n}(K); \Psi_{nm})$. The first result of the paper is that if the embeddings Ψ_{nm} satisfy certain natural dimensionality conditions and if $\mathcal{S} = \varinjlim (T_{p_n}(K); \Phi_{nm})$ is an arbitrary TUHF K -algebra, then $\mathcal{S}$ is K -isomorphic to $\mathcal{T}$, only if the supernatural number, $N[(p_n)]$, of (p_n) is less than or equal to the supernatural number, $N[(q_n)]$, of (q_n) . Thus, if the embeddings Φ_{nm} also satisfy the above dimensionality conditions, then $\mathcal{S}$ is K -isomorphic to $\mathcal{T}$, only if $N[(p_n)] = N[(q_n)]$. The second result of the paper is a nontrivial “triangular” version of the fact that if p, q are positive integers, then there exists a unital K -monomorphism $\Phi: M_q(K) \rightarrow M_p(K)$, only if $q|p$. The first result of the paper depends directly on the second result.

1. INTRODUCTION

Let K be a fixed but arbitrary field. In this paper we investigate a certain class of K -algebras (i.e., algebras over K) called *triangular UHF (TUHF) K -algebras*. The main result of our investigation is that for a broad family of TUHF K -algebras, the *supernatural number* associated with these K -algebras can be used to compare the isomorphism classes of these K -algebras. The main results are stated in Theorem I and Lemma I. For historical remarks on the origins of the idea of a TUHF K -algebra, see the remarks following Lemma I. In order to present the details of Theorem I and Lemma I, we need the following definitions.

Definition 1.1. Let K be an arbitrary field. Let $\mathcal{A}$ be a unital K -algebra. A family of elements $\{e_{ij} | 1 \leq i \leq j \leq n\}$ in $\mathcal{A}$ is said to be a *triangular system of algebraic matrix units in $\mathcal{A}$* if, for $1 \leq i \leq j \leq n$, the following conditions are satisfied:

- (i) $e_{ij} \neq 0$;

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- (ii) $\{e_{ii} | 1 \leq i \leq n\}$ is an orthogonal family of idempotents such that $\sum_{i=1}^n e_{ii} = 1$;
 (iii) for $1 \leq i \leq p \leq n$ and $1 \leq q \leq j \leq n$ we have

$$e_{ip}e_{qj} = \begin{cases} e_{ij}, & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2. Let (p_n) be a sequence of positive integers, and let there be given a family $\{\Phi_{nm} | n \geq m\}$ of unital K -monomorphisms $\Phi_{nm}: T_{p_m}(K) \rightarrow T_{p_n}(K)$ such that $\Phi_{np}\Phi_{pm} = \Phi_{nm}$ whenever $m \leq p \leq n$, where $T_{p_n}(K)$ is the K -algebra of all $p_n \times p_n$ upper triangular matrices over K . A *triangular UHF (TUHF) K -algebra* is any K -algebra that is K -isomorphic to an algebraic inductive limit of the form $\mathcal{T} = \varinjlim (T_{p_n}(K); \Phi_{nm})$.

Definition 1.3. Let (p_n) be a sequence of positive integers. The *supernatural number of (p_n)* is the function $N[(p_n)]$, defined on the set of prime numbers, such that for any prime number x ,

$$N[(p_n)](x) = \sup\{m | \exists n: x^m | p_n\}.$$

We are now ready to present a detailed statement of Theorem I and Lemma I.

Theorem I. Let $\mathcal{S}$ and $\mathcal{T}$ be TUHF K -algebras given by

$$\mathcal{S} = \varinjlim (T_{p_n}(K); \Phi_{nm}), \quad \mathcal{T} = \varinjlim (T_{q_n}(K); \Psi_{nm}).$$

For each n , let $\{F_{ij}^{(n)} | 1 \leq i, j \leq q_n\}$ be the standard system of matrix units for $T_{q_n}(K)$. Assume that the following condition holds:

- (i) For $m \leq n$,

$$\text{rank } \Psi_{nm}(F_{ij}^{(m)}) = \text{rank } \Psi_{nm}(F_{uv}^{(m)}), \quad 1 \leq i \leq j \leq q_m, \quad 1 \leq u \leq v \leq q_m.$$

If $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ is a K -algebra isomorphism from $\mathcal{S}$ onto $\mathcal{T}$, then $N[(q_n)] \leq N[(p_n)]$.

Moreover, for each n , let $\{E_{ij}^{(n)} | 1 \leq i, j \leq p_n\}$ be the standard system of matrix units for $T_{p_n}(K)$. If the following condition also holds, then $\mathcal{S}$ is K -isomorphic to $\mathcal{T}$, only if $N[(p_n)] = N[(q_n)]$:

- (ii) For $m \leq n$

$$\text{rank } \Phi_{nm}(E_{ij}^{(m)}) = \text{rank } \Phi_{nm}(E_{uv}^{(m)}), \quad 1 \leq i \leq j \leq p_m, \quad 1 \leq u \leq v \leq p_m.$$

The conclusion of Theorem I is strong in the sense that the TUHF K -algebra $\mathcal{S}$ is completely arbitrary; moreover, it is not assumed, a priori, that $\mathcal{S}$ or $\mathcal{T}$ have “diagonals” of any sort. Some condition similar to the above dimensionality condition (i) seems necessary. To see this, let (p_n) and (q_n) be any two unbounded, increasing sequences of positive integers. For $m \leq n$, define the mapping $\Phi_{nm}: M_m(K) \rightarrow M_n(K)$ by $\Phi_{nm}(x) = \text{diag}(x, x_{mm}, \dots, x_{mm})$, $x = (x_{ij}) \in M_m(K)$. Then the TUHF K -algebras $\mathcal{S} = \varinjlim (T_{p_n}(K); \Phi_{p_np_m})$ and $\mathcal{T} = \varinjlim (T_{q_n}(K); \Phi_{q_nq_m})$ are isomorphic as K -algebras; but, clearly, there is not necessarily a nontrivial relation between $N[(p_n)]$ and $N[(q_n)]$.

Lemma I. *Let n, p, q be positive integers. Let $\{G_{ij} | 1 \leq i \leq j \leq p\}$ be a triangular system of algebraic matrix units in $M_n(K)$. Let $\mathcal{A}$ be the subalgebra of $M_n(K)$ generated by the G_{ij} . Let $\{F_{ij} | 1 \leq i \leq j \leq q\}$ be a family of matrices in $\mathcal{A}$ such that the following conditions hold:*

- (i) $\{F_{ii} | 1 \leq i \leq q\}$ is an orthogonal family of idempotents such that $\sum_{i=1}^q F_{ii} = 1$;
- (ii) for $1 \leq i \leq j \leq q$, $F_{ij} = F_{ii}F_{ij}F_{jj}$;
- (iii) for $1 \leq i \leq j \leq q$ and $1 \leq r \leq s \leq q$, $\text{rank}(F_{ij}) = \text{rank}(F_{rs})$.

Then $q|p$.

Theorem I can be derived from Lemma I in a few lines of simple reasoning. Thus, the bulk of the paper is devoted to the proof of Lemma I.

The proof of Lemma I has the virtue of being completely self-contained, and it is “elementary” in the sense that anyone with a knowledge of elementary matrix theory will be able to follow the proof.

Lemma I seems interesting in its own right—in fact, the author has used this lemma at a key point in [B2]. The lemma is a nontrivial “triangular” version of the fact that if p, q are positive integers, then there exists a unital K -monomorphism $\Phi: M_q(K) \rightarrow M_p(K)$, only if $q|p$.

A natural question to ask is whether or not the converse of Theorem I holds: If two triangular UHF K -algebras have the same supernatural number, then are the algebras necessarily K -isomorphic to each other? The answer is “No.” To see this, let p, q be positive integers such that $q|p$ and define $d = p/q$. Define mappings $\rho_{pq}, \sigma_{pq}: M_q(K) \rightarrow M_p(K)$ by $\rho_{pq}(x) = x \otimes 1_d$, $\sigma_{pq}(x) = 1_d \otimes x$, $x \in M_q(K)$. Note that ρ_{pq}, σ_{pq} map $T_q(K)$ into $T_p(K)$. Now let (p_n) be an unbounded sequence of positive integers such that $p_m|p_n$ whenever $m \leq n$. The refinement triangular UHF K -algebra determined by the sequence (p_n) is the algebra $\mathcal{R} = \varinjlim (T_{p_n}(K); \varinjlim \rho_{p_n p_m})$, and the standard triangular UHF K -algebra determined by the sequence (p_n) is the algebra $\Sigma = \varinjlim (T_{p_n}(K); \varinjlim \sigma_{p_n p_m})$. The algebras $\mathcal{R}, \Sigma$ satisfy all conditions of Theorem I, and these algebras have the same supernatural number, but $\mathcal{R}$ is not K -isomorphic to Σ . To prove this, define $E \in T_{p_1}(K)$ by $E = \text{diag}(1, 0, \dots, 0)$, and let e be the canonical image of E in $\mathcal{R}$. Then we have $(1 - e)xe = 0$ for all $x \in \mathcal{R}$. Now suppose that $\mathcal{R}$ is K -isomorphic to Σ via $\Phi: \mathcal{R} \rightarrow \Sigma$. Then, with $f = \Phi(e)$, we have $(1 - f)xf = 0$ for all $x \in \Sigma$. But then f lies in $\mathcal{D} = \bigcup \mathcal{D}_n$, where $\mathcal{D}_n$ is the canonical image of $D_{p_n}(K)$ in Σ . Here, $D_{p_n}(K)$ is the K -algebra of all $p_n \times p_n$ diagonal matrices over K . From this, it follows that $f = 0$ or 1 . But this contradicts the fact that f is nontrivial. Thus, $\mathcal{R}$ is not K -isomorphic to Σ .

There exist certain classes of triangular UHF K -algebras such that, within any of these classes, Theorem I and its converse hold. This is the content of the following theorem, which is a straightforward consequence of Theorem I: For each pair of positive integers p, q such that $q|p$, let there correspond a unital K -monomorphism $\Theta_{pq}: M_q(K) \rightarrow M_p(K)$ with the property that $\Theta_{pr}\Theta_{rq} = \Theta_{pq}$ whenever $q|r|p$. Let (p_n) and (q_n) be sequences of positive integers such that $p_m|p_n$ and $q_m|q_n$ whenever $m \leq n$. Finally, let $\mathcal{S}$ and $\mathcal{T}$ be the TUHF K -algebras defined by $\mathcal{S} = \varinjlim (T_{p_n}(K); \varinjlim \Theta_{p_n p_m})$, $\mathcal{T} = \varinjlim (T_{q_n}(K); \varinjlim \Theta_{q_n q_m})$. Then $\mathcal{S}$ is K -isomorphic to $\mathcal{T}$ if, and only if $N[(p_n)] = N[(q_n)]$. This the-

orem is a generalized, *purely algebraic* extension of the main result in [B1]. In particular, the theorem applies to the mappings ρ_{pq} and σ_{pq} , i.e., two refinement TUHF K -algebras are K -isomorphic if and only if they have the same supernatural number—ditto for standard TUHF K -algebras.

Throughout the remainder of this paper, all algebras, monomorphisms, and isomorphisms will be assumed to be K -algebras, K -monomorphisms, and K -isomorphisms.

Historical remarks. In the 1990 paper [B1], the author introduced a concrete class of nonselfadjoint operator algebras which are currently called *standard triangular UHF (TUHF) algebras*. The main result in [B1] is that two standard TUHF algebras are isometrically isomorphic if and only if they have the same supernatural number.

Standard triangular UHF algebras are special cases of what are called *triangular UHF algebras*. Triangular UHF algebras are defined as follows. Let $\mathcal{A} = \bigcup \mathcal{M}_n$ be a UHF C^* -algebra, where $(\mathcal{M}_n)$ is an increasing sequence of finite-dimensional C^* -subalgebras of $\mathcal{A}$ such that for each n , $\mathcal{M}_n$ contains the identity of $\mathcal{A}$ and where $\mathcal{M}_n$ is $*$ -isomorphic to M_{p_n} for some positive integer p_n . A unital Banach algebra $\mathcal{T} \subseteq \mathcal{A}$ is said to be a *triangular UHF (TUHF) algebra* if $\mathcal{T} = \bigcup \mathcal{T}_n$, where $(\mathcal{T}_n)$ is an increasing sequence of finite-dimensional Banach subalgebras of $\mathcal{A}$ such that for each n , $\mathcal{T}_n$ is a Banach subalgebra of $\mathcal{M}_n$ and the diagonal, $\mathcal{T}_n^* \cap \mathcal{T}_n$, of $\mathcal{T}_n$ is a maximal abelian self-adjoint subalgebra of $\mathcal{M}_n$. *Standard TUHF algebras* are those TUHF algebras such that the embedding of $\mathcal{M}_n$ into $\mathcal{M}_{n+1}$ has the special form

$$x \mapsto 1 \otimes x.$$

In [B2], the above definition of TUHF algebras is extended to the category of Banach algebras; these generalized TUHF algebras are called *triangular UHF (TUHF) Banach algebras*: A Banach algebra $\mathcal{T}$ is said to be a *triangular UHF (TUHF) Banach algebra* if $\mathcal{T} = \bigcup \mathcal{T}_n$, where $(\mathcal{T}_n)$ is an increasing sequence of finite-dimensional Banach subalgebras of $\mathcal{T}$ such that each $\mathcal{T}_n$ is generated, as a Banach algebra, by a triangular system of matrix units $\{e_{ij}^{(n)} \mid 1 \leq i \leq j \leq p_n\}$ in $\mathcal{T}$.

The idea of a TUHF K -algebra, where K is an arbitrary field, arose from the author's desire to convert the principal result in [B2] into a pure piece of algebra (to borrow a phrase used by Kaplansky [AAR]). In fact, Theorem I is a purely algebraic version of the main result in [B2]; and, as mentioned above, Theorem I can be used to derive a purely algebraic version of the main result in [B1].

We conclude our historical remarks by mentioning that the classification result in [B1] has been extended by Peters, Poon, and Wagner in [PPW] and by Power in [P1, P2].

This paper is organized as follows. In §2 we prove Lemma I in §3 we present the short proof of Theorem I.

2. PROOF OF LEMMA I

In this section we prove a series of lemmas which culminates in the proof of Lemma I. The first lemma of this section is indispensable in the proof of Lemma I.

Lemma 2.1. *Let n be a positive integer. Let A, X, Y be matrices in $T_n(K)$. Suppose that $A = XA = AY$. Then*

$$(1) \quad \text{rank}(A) = \text{rank}(\text{diag}(X)A \text{diag}(Y)).$$

Proof. Let $X_1 = \text{diag}(X)$, $Y_1 = \text{diag}(Y)$, and $A_1 = AY_1$. We claim that X_1 is injective on the range of A_1 , i.e., $\text{rank}(X_1 A_1) = \text{rank}(A_1)$. To prove this, let $x \in K^n$ be such that $X_1 A_1 x = 0$. Then, because $XA_1 = A_1$, we have

$$(2) \quad (X - X_1)A_1 x = A_1 x.$$

The matrix $X - X_1$ is nilpotent, therefore there exists a smallest nonnegative integer m such that $(X - X_1)^m A_1 x = 0$. We must have $m = 0$; for if $m > 0$, then, by (2), $(X - X_1)^{m-1} A_1 x = (X - X_1)^m A_1 x = 0$, which is a contradiction. It follows that $A_1 x = 0$. This proves the claim. Thus, $\text{rank}(X_1 A_1) = \text{rank}(A_1)$. Now, because $Y^t A^t = A^t$, we may apply the same argument to the pair A^t, Y_1^t ; thus, $\text{rank}(A) = \text{rank}(A^t) = \text{rank}(Y_1^t A^t) = \text{rank}(AY_1)$, and hence $\text{rank}(A) = \text{rank}(X_1 A Y_1)$. $\square$

The next lemma states that if $\{E_{ij} | 1 \leq i \leq j \leq p\}$, $E_{ij} \in M_{n_i, n_j}(K)$, are the blocks of a triangular system of algebraic matrix units in $M_n(K)$, where $n = \sum_{i=1}^p n_i$, then we can “diagonalize”, in a certain way, block row 1 of this system; and Lemma 2.3 states that for arbitrary $1 \leq i \leq p$ we can diagonalize, in a similar way, block row i of this system.

Lemma 2.2. *Let $(n_k)_{k=1}^p$ be a sequence of positive integers. For $1 \leq i \leq j \leq p$, let $0 \neq E_{ij} \in M_{n_i, n_j}(K)$. Assume that the following conditions hold:*

- (i) for $1 \leq i \leq p$, $E_{ii} = 1_{n_i}$;
- (ii) for $1 \leq i \leq k \leq j \leq p$, $E_{ik} E_{kj} = E_{ij}$.

Then there exists invertible matrices $M_1, \dots, M_p$ and a sequence $(d_k)_{k=1}^p$ of positive integers such that for $1 \leq j \leq p$, $M_j \in M_{n_j}(K)$ and

$$n_1 = d_1 \geq d_2 \geq \dots \geq d_p; \quad n_j \geq d_j; \quad M_1^{-1} E_{1j} M_j = \begin{pmatrix} 1_{d_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. For $p = 1$, take $M_1 = 1_{n_1}$, $d_1 = n_1$, then $M_1^{-1} E_{11} M_1 = 1_{n_1}$. So the proof is trivial for $p = 1$. Now suppose that the lemma is true for $p \geq 1$. Let $\{E_{ij} | 1 \leq i \leq j \leq p+1\}$ be a family of $n_i \times n_j$ matrices such that:

- (iii) $E_{ii} = 1_{n_i}$, $1 \leq i \leq p+1$;
- (iv) $E_{ik} E_{kj} = E_{ij}$, $1 \leq i \leq k \leq j \leq p+1$.

The family $\{E_{ij} | 1 \leq i \leq j \leq p\}$ satisfies (i) and (ii); hence, by induction hypothesis, there exist invertible matrices $N_j \in M_{n_j}(K)$, $1 \leq j \leq p$, and a sequence $(d_j)_{j=1}^p$ of positive integers such that

$$(3) \quad n_1 = d_1 \geq d_2 \geq \dots \geq d_p; \quad n_j \geq d_j; \\ N_1^{-1} E_{1j} N_j = \begin{pmatrix} 1_{d_j} & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 \leq j \leq p.$$

Define $N_{p+1} = 1_{n_{p+1}}$, and let F_{ij} , $1 \leq i \leq j \leq p+1$, be defined by

$$F_{jp+1} = N_j^{-1} E_{jp+1}, \quad 1 \leq j \leq p+1; \quad F_{ij} = N_i^{-1} E_{ij} N_j, \quad 1 \leq i \leq j \leq p.$$

Then we have

- (v) $F_{ii} = 1_{n_i}$, $1 \leq j \leq p+1$;
- (vi) $F_{ik} F_{kj} = F_{ij}$, $1 \leq i \leq k \leq j \leq p+1$.

Because F_{pp+1} is an $n_p \times n_{p+1}$ matrix and because $d_p \leq n_p$, we can write $F_{pp+1} = \begin{pmatrix} X \\ Y \end{pmatrix}$, where X is a $d_p \times n_{p+1}$ matrix. We then have

$$(4) \quad 0 \neq F_{1p+1} = F_{1p}F_{pp+1} = N_1^{-1}E_{1p}N_p \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix}.$$

We conclude that $X \neq 0$. A result from matrix algebra [BL, Theorem 10.6] allows us to find a $d_p \times d_p$ matrix M and an $n_{p+1} \times n_{p+1}$ matrix N such that

$$(5) \quad M^{-1}XN \begin{pmatrix} 1_{d_{p+1}} & 0 \\ 0 & 0 \end{pmatrix}, \quad d_{p+1} = \min\{d_p, n_{p+1}\}.$$

Define M_i , $1 \leq i \leq p+1$, by

$$M_{p+1} = N; \quad M_i = N_i \begin{pmatrix} M & 0 \\ 0 & 1_{n_i-d_p} \end{pmatrix}, \quad 1 \leq i \leq p.$$

Note that the matrices M_i are well defined, because, by (3), $n_i \geq d_p$ for $1 \leq i \leq p$. For $1 \leq j \leq p$, we have

$$(6) \quad \begin{aligned} M_1^{-1}E_{1j}M_j &= \begin{pmatrix} M^{-1} & 0 \\ 0 & 1_{n_1-d_p} \end{pmatrix} \begin{pmatrix} 1_{d_p} & 0 & 0 \\ 0 & 1_{d_j-d_p} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & 1_{n_j-d_p} \end{pmatrix} \\ &= \begin{pmatrix} 1_{d_j} & 0 \\ 0 & 0 \end{pmatrix}; \end{aligned}$$

here, we have used the fact that, by (3), $n_1 \geq d_j$. By (4) and (5),

$$(7) \quad M_1^{-1}E_{1p+1}M_{p+1} = \begin{pmatrix} 1_{d_{p+1}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Combining conditions (4)–(7), we see that for $1 \leq j \leq p+1$

$$n_1 = d_1 \geq \cdots \geq d_{p+1}; \quad n_j \geq d_j; \quad M_1^{-1}E_{1j}M_j = \begin{pmatrix} 1_{d_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

This completes the inductive step from p to $p+1$; hence, the lemma is proved. $\square$

Lemma 2.3. *Let $(n_i)_{i=1}^p$ be a sequence of positive integers. Let $\{E_{ij} | 1 \leq i \leq j \leq p\}$ be a family of nonzero matrices over K such that $E_{ij} \in M_{n_i, n_j}(K)$ and the E_{ij} satisfy conditions (i) and (ii) of Lemma 2.2. Let $1 \leq i \leq p$ be arbitrary. Then there exist invertible matrices $N_j \in M_{n_j}(K)$, $1 \leq j \leq p$, and a sequence of positive integers $(m_j)_{j=1}^p$ such that for $1 \leq i \leq j \leq p$,*

$$n_i = m_i \geq m_j \geq m_{j+1} \geq \cdots \geq m_p; \quad n_j \geq m_j; \quad N_i^{-1}E_{ij}N_j = \begin{pmatrix} 1_{m_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. For $1 \leq j \leq k \leq p-i+1$, define n'_j and $F_{jk} \neq 0$ by

$$n'_j = n_{i+(j-1)}, \quad F_{jk} = E_{i+(j-1), i+(k-1)}.$$

Now apply Lemma 2.2 to the sequence $(n'_j)_{j=1}^{p-i+1}$ and the family $\{F_{jk} | 1 \leq j \leq k \leq p-i+1\}$. $\square$

Lemmas 2.4 and 2.6 are technical lemmas whose proofs have been separated from the proof of Lemma I. Lemma 2.5, which is presented without proof, is a simple but useful result from matrix algebra.

Lemma 2.4. Let $(n_i)_{i=1}^p$ be a sequence of positive integers. Let $\{E_{ij} | 1 \leq i \leq j \leq p\}$ be a family of nonzero matrices over K such that $E_{ij} \in M_{n_i, n_j}(K)$. Assume that the following conditions hold:

- (i) $E_{ii} = 1_{n_i}$, $1 \leq i \leq p$;
- (ii) $E_{ik}E_{kj} = E_{ij}$, $1 \leq i \leq k \leq j \leq p$.

Define $n = \sum_{i=1}^p n_i$. For $1 \leq i, j \leq p$, let $G_{ij} \in T_n(K)$ be the block upper triangular matrix such that for $1 \leq d \leq k \leq p$, the d - k block entry is $0_{n_d \times n_k}$, if $(d, k) \neq (i, j)$, and the i - j entry of G_{ij} is E_{ij} . Let $0 \neq H \in M_n(K)$ have the form

$$H = \sum_{1 \leq i \leq j}^p h_{ij} G_{ij},$$

and suppose that the following conditions hold:

- (iii) For $1 \leq i \leq p$, define $G_i = G_{ii}$. There exist sequences of positive integers

$$1 \leq r_1 < \cdots < r_d \leq p; \quad 1 \leq s_1 < \cdots < s_m \leq p$$

such that $EF = 0$ and $H = EH = HF$, where

$$E = \sum_{k=1}^d G_{r_k}, \quad F = \sum_{k=1}^m G_{s_k}.$$

(iv) For some pair $1 \leq u \leq d$, $1 \leq v \leq m$ of positive integers, $r_u \leq s_v$ and the matrix $h_{r_u s_v} E_{r_u s_v}$ is the first nonzero block entry in block row r_u and the last nonzero block entry in block column s_v .

- (v) $\text{rank}(H) = \sum_{k=1}^d n_{r_k} = \sum_{k=1}^m n_{s_k}$, where $n_i = \dim(G_i)$, $1 \leq i \leq p$.

Then there exists a matrix $J \in T_n(K)$ of the form $J = \sum_{1 \leq i \leq j}^p g_{ij} G_{ij}$ such that the following conditions are satisfied:

- (vi) $J = EJ = JF$;
- (vii) the matrix $g_{r_u s_v} E_{r_u s_v}$ is the only nonzero block entry in block row r_u and the only nonzero block entry in block column s_v ;
- (viii) $\text{rank}(J) = \text{rank}(H)$.

Proof. By Lemma 2.3, there exist invertible matrices $N_j \in M_{n_j}(K)$, $1 \leq j \leq p$, and a sequence of positive integers $(q_j)_{j=r_u}^p$ such that for $r_u \leq j \leq p$,

$$(8) \quad n_{r_u} = q_{r_u} \geq q_j \geq q_{j+1} \geq \cdots \geq q_p; \quad n_j \geq q_j; \quad N_{r_u}^{-1} E_{r_u j} N_j = \begin{pmatrix} 1_{q_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

Define $N \in M_n(K)$ by $N = \text{diag}(N_1, \dots, N_p)$. For $1 \leq i \leq j \leq p$, define F_{ij} , H_{ij} by

$$F_{ij} = N_i^{-1} E_{ij} N_j, \quad H_{ij} = N^{-1} G_{ij} N;$$

then it is easy to see that for $1 \leq d, k \leq p$, the d - k block entry of H_{ij} is $0_{n_d \times n_k}$, if $(d, k) \neq (i, j)$, and the i - j block entry of H_{ij} is F_{ij} ; so, $H_{ij} \in T_n(K)$. Define the matrices $L, P, Q \in T_n(K)$ by

$$L = N^{-1} H N = \sum_{1 \leq i \leq j}^p h_{ij} H_{ij}, \quad P = N^{-1} E N, \quad Q = N^{-1} F N.$$

We then have

- (ix) $L = PL = LQ$;

- (x) $F_{ii} = 1_{n_i}$, $1 \leq i \leq p$;
- (xi) $F_{ik}F_{kj} = F_{ij}$, $1 \leq i \leq k \leq j \leq p$;
- (xii) $h_{r_u s_v} F_{r_u s_v}$ is the first nonzero block entry in block row r_u and the last nonzero block entry in block column s_v ;
- (xiii) $\text{rank}(L) = \text{rank}(H)$.

Let $1 \leq i < r_u$ and $s_v < j \leq p$, and write

$$F_{i r_u} = \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ n_{r_u}}^{(i)} \\ \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ n_{r_u}}^{(i)} \end{pmatrix};$$

by (8), $n_{r_u} = q_{r_u} \geq q_{s_v} \geq q_j$, consequently,

$$\begin{aligned} (9) \quad F_{ij} &= F_{i r_u} F_{r_u j} = \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ n_{r_u}}^{(i)} \\ \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ n_{r_u}}^{(i)} \end{pmatrix} \begin{pmatrix} 1_{q_j} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ q_j}^{(i)} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ q_j}^{(i)} & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

It follows from (9) that for $1 \leq i < r_u$ and $s_v < j \leq p$, the entries $\begin{smallmatrix} (i, s_v) & (i, j) \\ (r_u, s_v) & (r_u, j) \end{smallmatrix}$ of L can be expressed as follows:

$$\begin{aligned} (10) \quad h_{i s_v} &\begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ q_{s_v}}^{(i)} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ q_{s_v}}^{(i)} & 0 & \cdots & 0 \end{pmatrix}, \quad h_{ij} \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ q_j}^{(i)} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ q_j}^{(i)} & 0 & \cdots & 0 \end{pmatrix}, \\ h_{r_u s_v} &\begin{pmatrix} 1_{q_{s_v}} & 0 \\ 0 & 0 \end{pmatrix}, \quad h_{r_u j} \begin{pmatrix} 1_{q_j} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Because $h_{r_u s_v} F_{r_u s_v}$ is the first nonzero block entry in block row r_u , and because $1 \leq i < r_u$ and $s_v < j \leq p$ are arbitrary, it follows from (xii) and (10) that by performing elementary row operations on L we can transform L into a matrix of the form $L' = \sum_{1 \leq d \leq k} h'_{dk} H_{dk}$ which has the property that $h'_{r_u s_v} F_{r_u s_v}$ is the first nonzero block entry in block row r_u and the only nonzero block entry in block column s_v , and $L' = PL' = L'Q$. Because L' is row equivalent to L , it follows from (xiii) that

$$(11) \quad \text{rank}(L') = \text{rank}(H).$$

By (9), if $1 \leq i < r_u$ and $s_v < j \leq p$, then the entries $\begin{smallmatrix} (i, s_v) & (i, j) \\ (r_u, s_v) & (r_u, j) \end{smallmatrix}$ of L' can be expressed as follows:

$$(12) \quad h'_{i \ s_v} \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ q_{s_v}}^{(i)} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ q_{s_v}}^{(i)} & 0 & \cdots & 0 \end{pmatrix}, \quad h'_{i \ j} \begin{pmatrix} a_{1 \ 1}^{(i)} & \cdots & a_{1 \ q_j}^{(i)} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_i \ 1}^{(i)} & \cdots & a_{n_i \ q_j}^{(i)} & 0 & \cdots & 0 \end{pmatrix},$$

$$h'_{r_u \ s_v} \begin{pmatrix} 1_{q_{s_v}} & 0 \\ 0 & 0 \end{pmatrix}, \quad h'_{r_u \ j} \begin{pmatrix} 1_{q_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

For $s_v < j \leq p$, (8) implies that $q_j \leq q_{s_v}$; therefore, because $h'_{r_u \ s_v} F_{r_u \ s_v}$ is the first nonzero block entry in block row r_u and the only nonzero block entry in block column s_v , and $L' = PL' = L'Q$, it follows from (12) that by performing elementary column operations on L' we can transform L' into a matrix of the form $M = \sum_{1 \leq i \leq j} q_{ij} H_{ij}$ which has the property that $g_{r_u \ s_v} F_{r_u \ s_v}$ is the only nonzero block entry in block row r_u and the only nonzero block entry in block column s_v , and $M = PM = MQ$. Because M is column equivalent to L' , (11) implies that

$$(13) \quad \text{rank}(M) = \text{rank}(H).$$

Now define $J = NMN^{-1} = \sum_{1 \leq i \leq j} g_{ij} G_{ij}$. Then J satisfies conditions (vi) and (vii) above; moreover, (13) implies that J satisfies condition (viii). Hence we have completed the proof of the lemma. $\square$

Lemma 2.5. *Let i, j, n be positive integers. Let $X, Y, Z \in M_n(K)$. Suppose that*

$$X = Y + Z, \quad \text{rank}(X) = i + j, \quad \text{rank}(Y) \leq i, \quad \text{rank}(Z) \leq j.$$

Then $\text{rank}(Y) = i$ and $\text{rank}(Z) = j$.

Lemma 2.6. *Let $(n_i)_{i=1}^p$ be a sequence of positive integers. Let $\{E_{ij} | 1 \leq i \leq j \leq p\}$ be a family of nonzero matrices over K such that $E_{ij} \in M_{n_i, n_j}(K)$. Assume that the following conditions hold:*

- (i) $E_{ii} = 1_{n_i}$, $1 \leq i \leq p$;
- (ii) $E_{ik} E_{kj} = E_{ij}$, $1 \leq i \leq k \leq j \leq p$.

Define $n = \sum_{i=1}^p n_i$. For $1 \leq i, j \leq p$, let $G_{ij} \in T_n(K)$ be the block upper triangular matrix defined in Lemma 2.4. Assume that there exists $0 \neq H \in M_n(K)$ of the form $H = \sum_{1 \leq i \leq j} h_{ij} G_{ij}$ with the following properties:

- (iii) *There exist sequences of positive integers,*

$$1 \leq r_1 < \cdots < r_d \leq p, \quad 1 \leq s_1 < \cdots < s_m \leq p,$$

such that $EF = 0$ and $H = EH = HF$, where $E = \sum_{k=1}^d G_{r_k}$, $F = \sum_{k=1}^m G_{s_k}$;

- (iv) $\text{rank}(H) = \sum_{k=1}^d n_{r_k} = \sum_{k=1}^m n_{s_k}$.

Then $d = m$.

Proof. We will prove the lemma by induction on d . Assume, first, that $d = 1$. By (iii), we have

$$H = EH = G_{r_1} H = G_{r_1} \sum_{1 \leq i \leq j} h_{ij} G_{ij} = \sum_{j \geq r_1} h_{r_1 j} G_{r_1 j} = \sum_{\substack{1 \leq k \leq m \\ s_k \geq r_1}} G_{r_1 s_k}.$$

Because $H \neq 0$, there is a smallest positive integer $1 \leq v \leq m$ such that $s_v \geq r_1$ and $h_{r_1 s_v} \neq 0$; we then have

$$H = \sum_{j=s_v}^p h_{r_1 j} G_{r_1 j}.$$

For $s_v \leq j \leq p$, we have

$$h_{r_1 j} G_{r_1 j} = (h_{r_1 s_v}^{-1} h_{r_1 j})(h_{r_1 s_v} G_{r_1 s_v}) G_{s_v j};$$

hence, $h_{r_1 j} G_{r_1 j}$ can be obtained from $h_{r_1 s_v} G_{r_1 s_v}$ by elementary column operations; therefore, H is column equivalent to $J = h_{r_1 s_v} G_{r_1 s_v}$, so $\text{rank}(H) = \text{rank}(J)$. From the definition of $G_{r_1 s_v}$, we have

$$\text{rank}(h_{r_1 s_v} G_{r_1 s_v}) = \text{rank}(E_{r_1 s_v}) \leq n_{s_v}.$$

Therefore, by (iv), $\text{rank}(H) = \sum_{k=1}^m n_{s_k} \leq n_{s_v}$, from which it follows that $m = 1 = d$. This proves the lemma for the case when $d = 1$.

Now suppose that the lemma is true for $d > 1$. Thus, assume that we are given a matrix $0 \neq H = \sum_{1 \leq i \leq j}^p h_{ij} G_{ij}$ with the following properties:

(v) There exist sequences of positive integers,

$$1 \leq r_1 < \cdots < r_{d+1} \leq p, \quad 1 \leq s_1 < \cdots < s_m \leq p,$$

such that $EF = 0$ and $H = EH = HF$, where $E = \sum_{k=1}^{d+1} G_{r_k}$, $F = \sum_{k=1}^m G_{s_k}$;

(vi) $\text{rank}(H) = \sum_{k=1}^{d+1} n_{r_k} = \sum_{k=1}^m n_{s_k}$.

Because $H \neq 0$, there exist $1 \leq u \leq d+1$ and $1 \leq v \leq m$ such that $r_u \leq s_v$ and $h_{r_u s_v} E_{r_u s_v}$ is the first nonzero block entry in block row r_u and the last nonzero block entry in block column s_v ; hence, by (v) and (vi), we may apply Lemma 2.4 to obtain a matrix $J \in T_n(K)$ of the form $J = \sum_{1 \leq i \leq j}^p g_{ij} G_{ij}$ which has the following properties:

(vii) The matrix $g_{r_u s_v} E_{r_u s_v}$ is the only nonzero block entry in block row r_u and the only nonzero block entry in block column s_v ;

(viii) $J = EJ = JF$;

(ix) $\text{rank}(J) = \sum_{k=1}^{d+1} n_{r_k} = \sum_{k=1}^m n_{s_k}$.

By (vii), $G_{r_u} J = J G_{s_v} = g_{r_u s_v} G_{r_u s_v}$. Therefore, by (viii),

$$J = EJF = \left(\sum_{k \neq u}^{d+1} G_{r_k} + G_{r_u} \right) J \left(\sum_{k \neq v}^m G_{s_k} + G_{s_v} \right) = J_1 + J_2;$$

where $J_1 = (\sum_{k \neq u}^{d+1} G_{r_k}) J (\sum_{k \neq v}^m G_{s_k})$ and $J_2 = g_{r_u s_v} G_{r_u s_v}$. Now, by (ix), we have

$$J = J_1 + J_2,$$

$$\text{rank}(J) = \sum_{k \neq u}^{d+1} n_{r_k} + n_{r_u}, \quad \text{rank}(J_1) \leq \sum_{k \neq u}^{d+1} n_{r_k}, \quad \text{rank}(J_2) = \text{rank}(G_{r_u s_v}) \leq n_{r_u};$$

therefore, by Lemma 2.5, $\text{rank}(J_1) = \sum_{k \neq u}^{d+1} n_{r_k}$. Likewise, $\text{rank}(J_1) = \sum_{k \neq v}^m n_{s_k}$. The matrix J_1 clearly has the form $J_1 = \sum_{1 \leq i \leq j}^p f_{ij} G_{ij}$. Define $E_1 = \sum_{k \neq u}^{d+1} G_{r_k}$, $F_1 = \sum_{k \neq v}^m G_{s_k}$. Then

$$\begin{aligned}
 J_1 &= \sum_{1 \leq i \leq j}^p f_{ij} G_{ij}, \\
 E_1 F_1 &= 0, \quad J_1 = E_1 J_1 = J_1 F_1, \\
 (14) \quad \text{rank}(J_1) &= \sum_{k \neq u}^{d+1} n_{r_k} = \sum_{k \neq v}^m n_{s_k}, \\
 1 \leq r_1 &< \cdots < r_{u-1} < r_{u+1} < \cdots < r_{d+1} \leq p, \\
 1 \leq s_1 &< \cdots < s_{v-1} < s_{v+1} < \cdots < s_m \leq p.
 \end{aligned}$$

Now apply the induction hypothesis to (14) to conclude that $d = m - 1$, i.e., $d + 1 = m$. This completes the proof of the lemma. $\square$

We now have enough machinery to prove the main lemma of this section.

Proof of Lemma I. Let n, p, q be positive integers, and let $\{G_{ij} | 1 \leq i \leq j \leq p\}$, $\mathcal{A}$, and $\{F_{ij} | 1 \leq i \leq j \leq q\}$ be as in the hypothesis of Lemma I. Lemma I is clearly true for $p = 1$, so we may assume that $p > 1$. For $1 \leq i \leq p$, define $G_i = G_{ii}$, $n_i = \dim(G_i)$. By simultaneously diagonalizing the family $\{G_{ii} | 1 \leq i \leq p\}$, we may assume, without loss of generality, that each G_i has the form

$$G_i = \text{diag}(0, \dots, 0, 1_{n_i}, 0, \dots, 0),$$

where the matrix 1_{n_i} is the i th block diagonal entry of G_i . Then there exists a family $\{H_{ij} | 1 \leq i \leq j \leq p\}$ of matrices over K such that for $1 \leq i \leq j \leq p$, $H_{ij} \in M_{n_i, n_j}(K)$, and such that the following conditions hold:

- (i) $H_{ii} = 1_{n_i}$, $1 \leq i \leq p$;
- (ii) $H_{ik} H_{kj} = H_{ij}$, $1 \leq i \leq k \leq j \leq p$;
- (iii) for $1 \leq i \leq j \leq p$, G_{ij} is the block upper triangular matrix in $T_n(K)$ such that for $1 \leq d, k \leq p$, the d - k entry of G_{ij} is $0_{n_d \times n_k}$, if $(d, k) \neq (i, j)$, and the i - j entry of G_{ij} is H_{ij} .

For $1 \leq i \leq q$, define $F_i = F_{ii}$; then because F_i is an idempotent in $\mathcal{A}$, it is easy to see that there corresponds to i a subsequence $(r_{ik})_{k=1}^{m_i}$ of $1, 2, \dots, p$ such that

$$\text{diag}(F_i) = \sum_{k=1}^{m_i} G_{r_{ik}}.$$

Fix $1 \leq i < j \leq q$. We have $F_i, F_j, F_{ij} \in T_n(K)$, with $F_{ij} = F_i F_{ij} F_j$; hence, by Lemma 2.1,

$$(15) \quad \text{rank}(F_{ij}) = \text{rank}(\text{diag}(F_i) F_{ij} \text{diag}(F_j)).$$

Define $E = \text{diag}(F_i)$, $F = \text{diag}(F_j)$, and $H = \text{diag}(F_i) F_{ij} \text{diag}(F_j)$. The matrix H clearly has the form $\sum_{1 \leq d \leq k}^p h_{dk} G_{dk}$. Then $EF = 0$ and $H = EH = HF$. By (15), $\text{rank}(H) = \text{rank}(\bar{F}_{ij}) = \dim(F_i) = \dim(F_j)$. We conclude that for

$$1 \leq i \leq j \leq p,$$

$$(16) \quad \begin{aligned} H &= \sum_{1 \leq d \leq k}^p h_{dk} G_{dk}, \\ EF &= 0, \quad H = EH = HF, \\ \text{rank}(H) &= \sum_{k=1}^{m_i} n_{r_{ik}} = \sum_{k=1}^{m_j} n_{r_{jk}}, \\ 1 &\leq r_{i1} < \cdots < r_{im_i} \leq p, \\ 1 &\leq r_{j1} < \cdots < r_{jm_j} \leq p. \end{aligned}$$

Conditions (i)–(iii) above allow us to apply Lemma 2.6 to (16), and we conclude that $m_i = m_j$. Because $1 \leq i < j \leq q$ are arbitrary, we have that for some positive integer d , $d = m_i$ for $1 \leq i \leq q$. Therefore,

$$1 = \sum_{i=1}^p G_i = \sum_{i=1}^q \text{diag}(F_i) = \sum_{i=1}^q \left(\sum_{k=1}^{m_i} G_{r_{ik}} \right) = \sum_{i=1}^q \left(\sum_{k=1}^d G_{r_{ik}} \right).$$

It follows that $p = dq$. This completes the proof of the lemma. $\square$

3. PROOF OF THEOREM I

In this final section we prove Theorem I.

Proof of Theorem I. Let $\mathcal{S} = \varinjlim (T_{p_n}(K); \Phi_{nm})$ and $\mathcal{T} = \varinjlim (T_{q_n}(K); \Psi_{nm})$ be the TUHF K -algebras given in Theorem I. Assume that the mappings Ψ_{nm} satisfy condition (i) of Theorem I. Suppose that $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ is an isomorphism from $\mathcal{S}$ onto $\mathcal{T}$. We want to show that $N[(q_n)] \leq N[(p_n)]$. To prove this, it will suffice to show that for each m , there exists an n such that $q_m | p_n$. So fix m . For each n , let Φ_n be the canonical embedding of $T_{p_n}(K)$ into $\mathcal{S}$ and let Ψ_n be the canonical embedding of $T_{q_n}(K)$ into $\mathcal{T}$. Then $\mathcal{S} = \bigcup \mathcal{S}_n$ and $\mathcal{T} = \bigcup \mathcal{T}_n$. We have $\mathcal{T} = \bigcup \Phi(\mathcal{S}_n)$, hence, there exists an n such that $\mathcal{T}_m \subseteq \Phi(\mathcal{S}_n)$. Because $\mathcal{T} = \bigcup \mathcal{T}_k$, there exists a $k \geq m$ such that $\Phi(\mathcal{S}_n) \subseteq \mathcal{T}_k$. For $1 \leq i \leq j \leq q_m$, define $f_{ij} = \Psi_{km}(F_{ij}^{(m)})$. For $1 \leq i \leq j \leq p_n$, define $h_{ij} = \Phi(\Phi_n(E_{ij}^{(n)}))$, $g_{ij} = \Psi_k^{-1}(h_{ij}) \in T_{q_k}(K)$. Let $\mathcal{A}$ be the subalgebra of $M_{q_k}(K)$ generated by g_{ij} , $1 \leq i \leq j \leq p_n$. Then we have $f_{ij} \in \mathcal{A}$ for $1 \leq i \leq j \leq q_m$. By assumption, we have

$$(17) \quad \text{rank}(f_{ij}) = \text{rank}(f_{uv}), \quad 1 \leq i \leq j \leq q_m, \quad 1 \leq u \leq v \leq q_m.$$

The families

$$\{f_{ij} | 1 \leq i \leq j \leq q_m\}, \quad \{g_{ij} | 1 \leq i \leq j \leq p_n\}$$

are triangular systems of algebraic matrix units in $M_{q_k}(K)$; therefore, by (17) and Lemma I, we conclude that $q_m | p_n$. Because m is arbitrary, this proves the first part of Theorem I. The last part of Theorem I is an immediate consequence of the first part. $\square$

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