SUBSERIES CONVERGENCE IN SPACES WITH A SCHAUDER BASIS

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ABSTRACT. Let E be a Hausdorff topological vector space having a Schauder basis $\{b_i\}$ and coordinate functionals $\{f_i\}$. Let $\sigma(E, F)$ be the weak topology on E induced by $F = \{f_i: i \in \mathbb{N}\}$. We show that if a series in E is subseries convergent with respect to $\sigma(E, F)$, then it is subseries convergent with respect to the original topology of E .

In [S] Stiles established what seems to be the first version of the Orlicz-Pettis Theorem for nonlocal spaces. Indeed, Kalton remarks in [K1] that Stiles's result motivated his far-reaching generalizations of the Orlicz-Pettis Theorem for series in a topological group [K2]. Stiles's version of the Orlicz-Pettis Theorem is for F -spaces with a Schauder basis, and his proof uses the metric properties of the space. Other proofs of Stiles's result have been given in [B] and [Sw1], and both of these proofs also rely heavily on the metric. Kalton has also given a substantial generalization of Stiles's result in the Corollary to Theorem 3 in [K2]; this generalization is also for metric linear spaces. In this note we will give a proof of Stiles's Orlicz-Pettis Theorem for arbitrary Hausdorff topological vector spaces which have a Schauder basis. Our proof uses a very interesting result of Antosik on the convergence of double series [A].

Throughout this note let E be a Hausdorff topological vector space with a Schauder basis $\{b_i\}$ and coordinate functionals $\{f_i\}$. Let $F = \{f_i: i \in \mathbb{N}\}$, and let $\sigma(E, F)$ be the weak topology on E from the duality between E and F . If τ is any vector topology on E , a series $\sum x_j$ in E is said to be τ subseries convergent if for every subsequence $\{x_{n_j}\}$ of $\{x_j\}$ the subseries $\sum x_{n_j}$ is τ convergent in E . We show that every series $\sum x_j$ in E which is $\sigma(E, F)$ subseries convergent is subseries convergent in the original topology of E .

For the proof of our result we use the following theorem of Antosik on iterated double series [A].

Theorem 1 (Antosik). *Let $x_{ij} \in E$ for $i, j \in \mathbb{N}$. If for every increasing sequence of positive integers $\{n_j\}$ the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{in_j}$ converges, then the double series $\sum_{i,j} x_{ij}$ converges. In particular, the iterated series $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij}$ converges and equals $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$.*

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Antosik proved Theorem 1 for series in a space equipped with a sequential convergence structure satisfying certain convergence properties [A, Theorem 3]. The theorem was established for series in an Abelian topological group in [Sw2]; the proof used only the Hahn-Schur Theorem for groups [AS, 8.1].

We now state and prove our generalization of Stiles's Orlicz-Pettis Theorem.

Theorem 2. *If $\sum x_j$ is $\sigma(E, F)$ subseries convergent in E , then $\sum x_j$ is subseries convergent in the original topology of E .*

Proof. We use Antosik's Theorem to show the convergence (in the original topology of E) of the double series $\sum_{i,j} \langle f_i, x_j \rangle b_i$. Let $\{n_j\}$ be an increasing sequence of positive integers, and let $x = \sum_{j=1}^{\infty} x_{n_j}$ be the $\sigma(E, F)$ sum of this series. For each i , $\sum_{j=1}^{\infty} \langle f_i, x_{n_j} \rangle b_i$ converges in E to $\langle f_i, x \rangle b_i$ since $\sum_{j=1}^{\infty} \langle f_i, x_{n_j} \rangle = \langle f_i, x \rangle$. Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle f_i, x_{n_j} \rangle b_i = \sum_{i=1}^{\infty} \langle f_i, x \rangle b_i = x.$$

By Theorem 1,

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle f_i, x_j \rangle b_i = \sum_{j=1}^{\infty} x_j$$

converges in the original topology of E . Since the same argument can be applied to any subseries of $\sum x_j$, the result follows.

The use of Antosik's Interchange Theorem in the proof of Theorem 2 removes both the metrizability and completeness assumptions from Stiles's Theorem [S]. Note that we did not assume the continuity of the coordinate functionals $\{f_i\}$ in our proof, so the topology $\sigma(E, F)$ and the original topology of E may not even be comparable.

The proof of Theorem 2 covers even more general situations than topological vector spaces with Schauder bases. We indicate an example of such an application.

Let (G, τ) be an abelian Hausdorff topological group, and assume there exists a sequence of homomorphisms $P_i: G \rightarrow G$ such that $\sum_{i=1}^{\infty} P_i x = x$ (convergence in τ) for each $x \in G$. In the case of E above, $P_i x = \langle f_i, x \rangle b_i$ is an example; if G is a vector space and each P_i is a projection, then a sequence $\{P_i\}$ satisfying the condition above is called a Schauder decomposition. Antosik's Interchange Theorem is valid for group-valued series [Sw2], and the proof of Theorem 2 gives

Theorem 3. *Let σ be a Hausdorff group topology on G , and assume that each P_i is $\sigma - \tau$ continuous. If $\sum x_i$ is σ subseries convergent, then $\sum x_i$ is τ subseries convergent.*

In Theorem 2, $P_i x = \langle f_i, x \rangle b_i$ is $\sigma(E, F) - \tau$ continuous, where τ is the original topology of E , so Theorem 2 is an immediate corollary of Theorem 3.

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