NONPOSITONE ELLIPTIC PROBLEMS IN $\mathbb{R}^n$

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(Communicated by Barbara Lee Keyfitz)

ABSTRACT. We consider the problem of the existence of positive decaying solutions in $\mathbb{R}^n$ to the nonlinear equation $lu = \lambda f(x, u)$ where l denotes a second-order uniformly elliptic operator and $f(x, u)$ is superlinear and subcritical with $f(x, 0) \leq 0$. Existence is obtained by Mountain Pass Arguments and positivity by establishing bounds for u in various Sobolev norms and by comparison with the case $l = -\Delta$.

1. INTRODUCTION

The theory of elliptic nonpositone problems has received considerable attention since its inception in the symmetry-breaking considerations of Smoller and Wasserman in the mid-1980s [19–21]. We recall in particular the papers of Allegretto, Nistri, Zecca, Castro, Shivaji, and Brown and the thesis of Unsurangsie ([1], [5–8] and [23]) where existence, nonexistence, and stability are discussed. If one considers also the more typical positone situation, then a great deal of work has been done by many mathematicians. A partial list of recent results is given by the papers of Bahri and Lions [4], Y. Y. Li [12], Rother [18], Lie and Ni [11], Bahri and Coron [3], P. L. Lions [13], Rabinowitz [17], Coti Zelati and Rabinowitz [9], and Noussair and Swanson [15]. The interested reader will find further references in these papers. The prototype equation considered is of the following type.

$$(1) \quad \begin{cases} lu \equiv -\sum D_i(a_{ij}(x)D_j u) = \lambda f(x, u) & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega \end{cases}$$

with l a uniformly elliptic operator in the smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and the problem is to obtain conditions for the existence of a positive solution. What distinguishes nonpositone problems given by (1) from the more typical nonlinear equation of the same form is the assumption: $f(x, 0) \leq 0$, $f(x, 0) \not\equiv 0$. To the best of our knowledge, all of the previous considerations have dealt with the case of a bounded domain Ω . We consider here the case $\Omega = \mathbb{R}^n$, $u \rightarrow 0$ at ∞ and, thus, must deal with the resulting lack of compactness of the embedding operators between the canonical Sobolev spaces. We shall

Received by the editors February 4, 1992 and, in revised form, May 20, 1993.

1991 *Mathematics Subject Classification.* Primary 35J60, 35J20.

Key words and phrases. Nonpositone, elliptic, positive decaying solution.

Research supported by NSERC (Canada).

assume that f is superlinear and subcritical at ∞ , consequently the existence of a solution u will follow by Mountain Pass arguments as in the positone case [2, 23]. For completeness and convenience in later proofs, we first briefly sketch how this is done. We next consider the question of showing the positivity of u and observe that we cannot use the Maximum Principle directly to guarantee $u > 0$. Instead we replace this with estimates on various norms of u in various subsets of Ω and with arguments based on the special case $a_{ij} = \delta_{ij}$, that is, for $l = -\Delta$. Theorems on Green's functions of Stampacchia [22] and Littman, Stampacchia, and Weinberger [14] are then recalled to deal with the general case. Illustrative examples conclude the paper.

2. ASSUMPTIONS AND PRELIMINARY RESULTS

We assume for simplicity that $a_{ij} \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$, $f \in C_{\text{loc}}^\alpha(\mathbb{R}^{n+1})$, and that the symmetric matrix (a_{ij}) satisfies $aI \leq (a_{ij}(x)) \leq bI$ for some positive constants α , a , b . The letters C , K , H denote generic constants whose value may change within the same proof, unless otherwise specified.

We recall the space E defined in [2] as the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_0^2 = \int_{\mathbb{R}^n} |\nabla u|^2 + \sigma u^2 dx$$

with $\sigma = (1 + |x|^2)^{-1}$ and further recall the observation that $\|u\|_0 \sim \|u\|_l$ where $\|u\|_l^2 = (lu, u)$ as a consequence of Hardy's Inequality.

We next observe that we may always write $f(x, t) \triangleq g(x, t) + f(x, 0)$ and take, without loss of generality, $g(x, t) \equiv 0$ if $t \leq 0$. As mentioned earlier, we deal with the subcritical case; specifically, we assume that:

- (I) $|g(x, t)| \leq f_1(x)t^s$, $|f(x, 0)| \leq f_0(x)$, with $t \geq 0$, $f_0 \in L_{\sigma^{-1}}^2 \cap L^\infty$, $f(x, 0) \leq 0$, $f_1 \in L^{p_0} \cap L^\infty(\mathbb{R}^n)$, $1 < s < \frac{n+2}{n-2}$, $p_0 = \frac{2n}{[2n-(s+1)(n-2)]}$.

Here $L_{\sigma^{-1}}^2$ denotes the L^2 space with weight σ^{-1} .

Further assumptions on f will be made below. We observe, however, that assumptions (3), (4) on f given in [2] do not hold here, due to the nonpositone nature of f . On the other hand, we are concerned only with positive solutions for λ small and not for every λ as was the case in [2].

We let: $G(u) = \int_{\mathbb{R}^n} F(x, u)$, $F(x, u) = \int_0^u f(x, \xi) d\xi$, $J(u) = \frac{1}{2}\|u\|_l^2 - \lambda G(u)$ for $u \in E$ and recall:

Lemma 0. (a) *Sobolev's Inequality:* $\|u\|_{(2n)/(n-2)} \leq c\|\nabla u\|_2$ holds in E .

(b) If $u \in E$, then u^+ and $u^- \in E$.

(c) G and J are weakly lower semicontinuous with $G'(u)(\varphi) = \int_{\mathbb{R}^n} f(x, u)\varphi$.

(d) G' is continuous and compact from E to E .

The explicit proof of these statements may be found, for example, in [2]. The weak solutions u of (1) in E may thus be described as the critical points of J , and we finally recall for such solutions:

Lemma 1. (a) $u \in L^p(\mathbb{R}^n)$, $(2n)/(n-2) \leq p \leq \infty$.

(b) $\lim_{|x| \rightarrow \infty} u = 0$.

Once again the proofs given in [2] still hold.

3. EXISTENCE

We first show that (1) has a solution and estimate its norm. Let us express $F(x, u) = F_1(x, u) + F_2(x, u)$ with

$$F_1(x, u) = \int_0^u g(x, t) dt \quad \text{and} \quad F_2(x, u) = f(x, 0)u.$$

We now assume that

- (II) $F_1(x, \alpha_0) > 0$ for some constant $\alpha_0 > 0$ and x in some open set Ω_0 and that for $\mu = s + 1$ we have $\mu F_1(x, t) \leq t g(x, t)$ for $x \in \mathbb{R}^n$, $t \geq 0$.

Theorem 1. *Under the above assumptions (I), (II) on f , problem (1) has for λ small a solution $u \in E$ such that*

$$K\lambda^{-1/(s-1)} \leq \|u\|_l \leq H\lambda^{-1/(s-1)},$$

for some constants K, H independent of u .

Proof. The proof is an adaptation of ideas from [2] and [23] and basically is an application of the Mountain Pass Theorem [16]. We first observe that the PS condition is satisfied. Indeed, assume $\{u_m\}$ is a sequence with $\{J(u_m)\}$ bounded and $J'(u_m) \rightarrow 0$. In view of Lemma 0(a)–(d) and [16] it suffices to show that $\|u_m\|_l$ is bounded, and to see this, we observe:

$$\begin{aligned} J'(u_m)(u_m^-) &= - \int_{\mathbb{R}^n} \sum a_{ij}(u_m^-)_i (u_m^-)_j - \lambda \int_{\mathbb{R}^n} (g(x, u_m)u_m^- + f(x, 0)u_m^-) \\ &= - \left[\int_{\mathbb{R}^n} \sum a_{ij}(u_m^-)_i (u_m^-)_j + \lambda f(x, 0)u_m^- \right] \end{aligned}$$

since $g(x, \xi) = 0$ if $\xi \leq 0$. We conclude that

$$\|u_m^-\|_l^2 \leq -J'(u_m)(u_m^-) + c\|u_m^-\|_l,$$

whence $\{u_m^-\}$ is bounded. Next,

$$J'(u_m)(u_m^+) = \int_{\mathbb{R}^n} \sum a_{ij}(u_m^+)_i (u_m^+)_j - \lambda \int_{\mathbb{R}^n} (g(x, u_m)u_m^+ + f(x, 0)u_m^+).$$

However,

$$K \geq J(u_m) = \frac{\|u_m\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} [F_1(x, u_m) + f(x, 0)u_m],$$

whence

$$\frac{\|u_m^+\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} F_1(x, u_m^+) \leq K$$

since $\{u_m^-\}$ is bounded and $f(x, 0) \leq 0$. Hence

$$\frac{\|u_m^+\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} \frac{g(x, u_m^+)u_m^+}{\mu} \leq K,$$

that is,

$$\frac{\|u_m^+\|_l^2}{2} + \frac{1}{\mu} [J'(u_m)(u_m^+) - \|u_m^+\|_l^2] + \frac{\lambda}{\mu} \int_{\mathbb{R}^n} f(x, 0)u_m^+ \leq K$$

and since $\mu > 2$,

$$\|u_m^+\|_l^2 - C_0 \|u_m^+\|_l \leq C_1 [K + |J'(u_m)(u_m^+)|]$$

for some constants C_0, C_1 . We conclude $\{u_m^+\}$ is bounded, and therefore so is $\{u_m\}$.

Next,

$$\begin{aligned} J(u) &= \frac{\|u\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} F_1(x, u) - \lambda \int_{\mathbb{R}^n} f(x, 0)u \\ &\geq \frac{1}{2} \|u\|_l^2 - \frac{\lambda}{s+1} \int_{\mathbb{R}^n} f_1 |u|^{s+1} - \lambda \int_{\mathbb{R}^n} f(x, 0)u \\ &\geq \frac{1}{2} \|u\|_l^2 - \frac{\lambda}{s+1} C \|u\|_l^{s+1} - \lambda K \|u\|_l \\ &= \|u\|_l \left[\frac{1}{2} \|u\|_l - \frac{\lambda}{s+1} C \|u\|_l^s - \lambda K \right] \end{aligned}$$

for some constants C, K . Choosing λ small then shows that $J(u) \geq \alpha > 0$ for $\|u\|_l = 1$.

Finally, we recall $F_1(x, \alpha_0) > 0$ in Ω_0 and choose any function $0 < \omega \in C_0^\infty(\Omega_0)$. Consider

$$J(t\omega) = \frac{t^2 \|\omega\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} F_1(x, t\omega) - \lambda t \int_{\mathbb{R}^n} f(x, 0)\omega.$$

As in [2] and elsewhere, we observe that in $\text{supp}(\omega)$, $F_1(x, t) \geq t^\mu a_1 - a_2$ for some constants a_1, a_2 , whence, since $\omega \in C_0^\infty$, we obtain

$$\begin{aligned} J(t\omega) &\leq \frac{t^2 \|\omega\|_l^2}{2} - \lambda \int_{\Omega_0} [(t\omega)^\mu a_1 - a_2] - \lambda t \int_{\mathbb{R}^n} f(x, 0)\omega \\ &\leq \frac{t^2 \|\omega\|_l^2}{2} - \lambda t^\mu \alpha_1 + \lambda(t+1)\alpha_2 \end{aligned}$$

for some constants α_1, α_2 . We conclude immediately that $J(t\omega) < 0$ for all t large and the existence of a critical point of J —that is, of a solution of (1)—follows by the Mountain Pass Theorem for all small $\lambda > 0$.

Finally, for this solution we have

$$\begin{aligned} \|u\|_l^2 &= \lambda \int_{\mathbb{R}^n} [g(x, u)u + f(x, 0)u] \leq \lambda \int_{\mathbb{R}^n} [f_1 |u|^{s+1} + f(x, 0)u] \\ &\leq \lambda G[\|f_1\|_{L^{p_0}} \|u\|_l^{s+1} + \|f(x, 0)\|_{L_{\sigma-1}^2} \|u\|_l]. \end{aligned}$$

From this inequality and the observation that $\|u\|_l$ must be bounded away from zero since $J(u)$ is, it immediately follows that $\|u\|_l \geq K\lambda^{-1/(s-1)}$. Also, let $\omega \in C_0^\infty$ be as above. We note:

$$\frac{dJ}{dt}(t\omega) = t\|\omega\|_l^2 - \lambda \int_{\mathbb{R}^n} g(x, t\omega)\omega - \lambda \int_{\mathbb{R}^n} f(x, 0)\omega.$$

But since on $\text{supp}(\omega)$, $t^\mu a_1 - a_2 \leq F_1(x, t) \leq \frac{1}{\mu} g(x, t)$, we have, for some constants, α, β, γ , and $t > 1$, that

$$\frac{dJ}{dt}(t\omega) \leq t\alpha - \lambda t^{\mu-1}\beta + \lambda\gamma.$$

We recall that $\mu = s + 1$. Hence if $t \geq C_1 \lambda^{-1/(s-1)}$, then for λ small enough (depending on ω) $\frac{dJ}{dt} \leq 0$. On the other hand, as shown earlier,

$$J(t\omega) \leq \frac{t^2}{2} \|\omega\|_l^2 - \lambda t^\mu \alpha_1 + \lambda(t+1)\alpha_2,$$

that is,

$$J(t\omega) \leq C \lambda^{-2/(s-1)} \frac{\|\omega\|_l^2}{2}.$$

We conclude $J(u) \leq \lambda^{-2/(s-1)} K$ by the characterization of the Mountain Pass Critical Point. Now

$$J(u) = \frac{\|u\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} F_1(x, u) - \lambda \int_{\mathbb{R}^n} f(x, 0)u,$$

that is,

$$J(u) \geq \frac{\|u\|_l^2}{2} - \lambda \int_{\mathbb{R}^n} f(x, 0)u - \frac{\|u\|_l^2}{\mu} + \lambda \int_{\mathbb{R}^n} \frac{f(x, 0)u}{\mu},$$

i.e.,

$$C \lambda^{-2/(s-1)} \geq \|u\|_l^2 - \lambda \alpha \|u\|_l,$$

and we obtain the estimate $\|u\|_l \leq H \lambda^{-1/(s-1)}$.

We shall establish the positivity of the solution by means of several lemmas. We introduce the following assumptions:

- (III) $f(x, 0)f_1^\delta \in L^1$ with $\delta = (n-2)/[2n-(s+1)(n-2)]$, $g(x, t) \geq 0$ for $t > 0$, $f_1 > 0$, and for any ball B there exists $\alpha_0 = \alpha_0(B)$ such that $\text{meas}(\{x | F_1(x, \alpha_0) = 0\} \cap B) = 0$.

Observe that assumption (I) implies, by Holder's Inequality, that $f(x, 0)f_1^\delta \in L^{1+\varepsilon}$ for any $\varepsilon > 0$. Consequently, this part of assumption (III) is only moderately more restrictive.

Lemma 2. Assume assumptions (I), (II), (III) hold. Then for λ , $\varepsilon > 0$ sufficiently small, there exists a ball B and constant C , independent of u such that

$$\text{meas}(\{x | u > \varepsilon f_1^\delta \lambda^{-1/(s-1)}\} \cap B) > C.$$

Proof. Let $S = \{x | |u(x)| > \varepsilon f_1^\delta \lambda^{-1/(s-1)}\}$ and observe that $\delta(s+1) = p_0 - 1$. Without loss of generality, we may assume f_1 is locally bounded below by a positive constant. By Theorem 1 we have the estimate

$$\begin{aligned} C_0 \lambda^{-2/(s-1)} &\leq \|u\|_l^2 = \lambda \int_S [g(x, u)u + f(x, 0)u] + \lambda \int_{\mathbb{R}^n - S} [g(x, u) + f(x, 0)]u \\ &\leq \lambda \int_S [f_1 |u|^{s+1} + |f(x, 0)| |u|] + \lambda \int_{\mathbb{R}^n - S} [f_1 |u|^{s+1} + |f(x, 0)| |u|] \\ &\leq C \lambda [\|f_1\|_{L^{p_0}(S)} \|u\|_l^{s+1} + \|f(x, 0)\|_{L^2_{\sigma^{-1}}(S)} \|u\|_l] \\ &\quad + \varepsilon^{s+1} \lambda \int_{\mathbb{R}^n - S} f_1^{p_0} \lambda^{-(s+1)/(s-1)} + \varepsilon \int_{\mathbb{R}^n - S} |f(x, 0)| f_1^\delta \lambda^{1-1/(s-1)} \\ &\leq C_1 [\|f_1\|_{L^{p_0}(S)} \lambda^{1-(s+1)/(s-1)} + \|f(x, 0)\|_{L^2_{\sigma^{-1}}(S)} \lambda^{1-1/(s-1)}] \\ &\quad + \lambda^{1-(s+1)/(s-1)} \varepsilon^{s+1} \|f_1\|_{L^{p_0}(\mathbb{R}^n)}^{p_0} + \int_{\mathbb{R}^n} |f(x, 0)| f_1^\delta \varepsilon \lambda^{1-1/(s-1)}, \end{aligned}$$

i.e.,

$$C_0 \leq C_1 [\|f_1\|_{L^{p_0}(S)} + \|f(x, 0)\|_{L^2_{\sigma^{-1}}(S)} \lambda^{s/(s-1)}] \\ + \varepsilon^{s+1} \|f_1\|_{L^{p_0}(\mathbb{R}^n)}^{p_0} + \varepsilon \lambda^{s/(s-1)} \|f(x, 0) f_1^\delta\|_{L^1(\mathbb{R}^n)}.$$

Choosing λ, ε sufficiently small, it follows that $\|f_1\|_{L^{p_0}(S)} \geq C_0/2$ with C_0 independent of u . Since $f_1 \in L^{p_0}(\mathbb{R}^n)$ by assumption, we conclude that there exists a ball $B \subset \mathbb{R}^n$ and constant C independent of u, λ and such that $\text{meas}(S \cap B) > C/4$, and the result follows.

Now let $0 < \omega \in E$ be such that

$$l\omega = -f(x, 0) \triangleq -\tilde{f}(x);$$

we then have

$$l(u + \lambda\omega) = \lambda g(x, u).$$

We do recall $g(x, t) = 0$ if $t \leq 0$, whence $u + \lambda\omega \geq 0$ and, in particular, $u^- \leq \lambda\omega$. It follows that for ε, λ small enough, $u > \varepsilon f_1^\delta \lambda^{-1/(s-1)}$ in $S \cap B$. Since $\mu F_1 \leq t g(x, t)$ implies $t g(x, t) \geq \mu F_1(x, \alpha_0) t^\mu \alpha_0^{-\mu}$ for $t > \alpha_0$, i.e., $g(x, t) \geq \mu t^s F_1(x, \alpha_0) \alpha_0^{-\mu}$ for x in $S \cap B, t > \alpha_0$. Therefore,

$$l(u + \lambda\omega) \geq \lambda \alpha F_1(x, \alpha_0) f_1^{\delta s} \lambda^{-s/(s-1)}|_{S \cap B} \\ \geq \alpha F_1(x, \alpha_0) f_1^{\delta s} \lambda^{-1/(s-1)}|_{S \cap B}$$

for λ small enough and some $\alpha > 0$, since $f_1 > 0$, bounded below in $S \cap B$.

Choose and fix any ball $B_1 \supset \bar{B}$. Note that $u + \lambda\omega \geq z$ where

$$l(z) = \alpha F_1(x, \alpha_0) f_1^{\delta s} \lambda^{-1/(s-1)}|_{S \cap B} \quad \text{in } B_1, \\ z = 0 \quad \text{on } \partial B_1$$

and hence

$$\inf_{\partial B} (u + \lambda\omega) \rightarrow +\infty \quad \text{as } \lambda \rightarrow 0,$$

since $F_1(x, \alpha_0) f_1^{\delta s}$ is nontrivial in S . Indeed, observe that if G_1 denotes the Green's function for B_1 then $G_1(x, y) > \pi > 0$ for some $\pi > 0$ and any $(x, y) \in \bar{B} \times \bar{B}$ [22]. It follows that $z(x) \geq \pi \alpha \lambda^{-1/(s-1)} \|F_1(\cdot, \alpha_0) f_1^{\delta s}\|_{L^1(S \cap B)}$, and we obtain the result.

Lemma 3. *There exists a function $C(\lambda)$ such that for $|x|$ near infinity*

$$(u + \lambda\omega)(x) \geq C(\lambda) \frac{1}{|x|^{n-2}},$$

and $C(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 0$.

Proof. Suppose first that $l = -\Delta$. Then this is immediate from the three-sphere theorem, since $l(u + \lambda\omega) \geq 0, u + \lambda\omega \geq 0$. To deal with the more general case, we basically repeat some of the arguments in [14] and consider a nested sequence of balls $\{B_i\}$ exhausting $\mathbb{R}^n$. Let G_i, S_i denote the Green's functions associated with l and $-\Delta$, respectively, with Dirichlet conditions on ∂B_i . Select a point x , and choose a ball B_0 centered at x . Note that $CS_i(x, y) \leq G_i(x, y) \leq DS_i(x, y)$ for any $y \in \partial B_0$ and i large enough with C, D constants independent of i, x, y , and the radius of B_0 [22, 14]. It follows that $u(x) + \lambda\omega(x) \geq C\tau_i(x)$ where $-\Delta\tau_i = \alpha F_1(x, \alpha_0) f_1^{\delta s} \lambda^{-1/s-1}|_{S \cap B}, \tau_i = 0$ on ∂B_i . Passing to the limit as $i \rightarrow \infty$ gives the estimate.

To conclude the preliminary steps, we need only show that $\omega \leq \frac{d}{|x|^{n-2}}$ if $l\omega = -\tilde{f}(x)$ for some d . Recall $\tilde{f}(x) = f(x, 0)$ and that the arguments in Lemma 3 indicate that it suffices to show this for $l = -\Delta$. To see this, choose and fix a point x_0 and denote by ω_i the solution to $l(\omega_i) = -\tilde{f}|_{B_{\varepsilon i}(x_0)}$, $\omega_i = 0$ on $\partial B_i(x_0)$ for some small fixed ε , and with $B_r(x_0)$ denoting the sphere of radius r centered at x_0 . We then note that $\omega_i \uparrow \omega$ pointwise and $\omega_i \rightarrow \omega$ weakly in E , as $i \rightarrow \infty$. Let v_i, v represent the analogous functions to ω_i, ω with l replaced by $-\Delta$. Again from the results of [22, 10], we have $\omega_i(x_0) \leq Dv_i(x_0)$, with D independent of x_0, i . Passing to the limit, we obtain $\omega(x_0) \leq Dv(x_0)$.

Consequently assume $-\Delta\omega = -\tilde{f}(x)$. We immediately obtain from the Riesz Potential formula that

$$\omega(x) = T_1 \int_0^\infty \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt$$

and thus require that

$$(*) \quad |x|^{n-2} \int_0^\infty \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt \leq C$$

for $|x|$ large. It is easy to give sufficient conditions on $\tilde{f}(x)$ for this to be the case.

As a specific example we have:

Lemma 4. Assume $\tilde{f} \in L^1$ and $|x|^{n/q} \|\tilde{f}\|_{L^p(\mathbb{R}^n - B_{|x|}(0))} \leq C$ with $p > \frac{n}{2}$, $\frac{1}{q} + \frac{1}{p} = 1$; then $(*)$ holds.

Proof. We may write

$$\begin{aligned} \int_0^\infty \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt &= \int_{|x|/2}^\infty \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt + \int_0^{|x|/2} \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt \\ &= J_1 + J_2. \end{aligned}$$

Now $J_1 \leq C \|\tilde{f}\|_{L^1(\mathbb{R}^n)} |x|^{2-n}$ while

$$\begin{aligned} J_2 &= \int_0^{|x|/2} \frac{1}{t^{n-1}} \|\tilde{f}\|_{L^1(B_t(x))} dt \\ &\leq C \int_0^{|x|/2} t^{1-n+n/q} \|\tilde{f}\|_{L^p(B_t(x))} dt \\ &\leq C \|\tilde{f}\|_{L^p(B_{|x|/2}(x))} \left(\frac{|x|}{2} \right)^{2-n+n/q} \\ &\leq C \|\tilde{f}\|_{L^p(\mathbb{R}^n - B_{|x|/2}(0))} \left[\frac{|x|}{2} \right]^{n/q} \cdot |x|^{2-n} \end{aligned}$$

and the estimate follows.

Theorem 2. Assume $f_1 > 0$, $g(x, t) \geq 0$ if $t > 0$. Further assume that $f(x, 0) \equiv \tilde{f}(x)$ satisfies the conditions of Lemma 4 and that assumptions (I), (II), (III) hold. Then the solution u found in Theorem 1 is positive.

The proof of Theorem 2 follows immediately from Lemmas 1–4.

4. EXAMPLES

To illustrate the above results, and motivated by the positone cases considered in [2], we examine:

$$(4.1) \quad \begin{cases} -\Delta u = \lambda[g(x)u^s - h(x)] & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u = 0 \end{cases}$$

with $g, h > 0$, smooth, bounded $n > 2$, and $1 < s < \frac{n+2}{n-2}$.

Corollary 1. Assume $g \in L^{p_0}$ and $|x|h \in L^2$, $h \in L^1 \cap L^\infty$. Furthermore assume $|x|^{n/q} \|h\|_{L^p(\mathbb{R}^n - B_{|x|}(0))} \leq C$ for some $p > n/2$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then (4.1) has a positive decaying solution for λ small.

Proof. We need only observe that $h \in L^1 \cap L^\infty$ implies that $h \in L^p$ for any p , and in particular we conclude $hg^\delta \in L^1$ by Hölder's Inequality, as $\delta < p_0$. Finally note that $|x|h \in L^2$ and $h \in L^1$ imply $h \in L^2_{\sigma^{-1}}$ and all the conditions of Theorem 2 hold.

If we further assume g, h are radial functions, i.e., $g = g(|x|)$, $h = h(|x|)$, then the conditions of Corollary 1 hold immediately if $h(|x|) = O(|x|^{-b})$ and $g(|x|) = O(|x|^{-a})$ at infinity, for $b > n$ and $[2n - (s+1)(n-2)] < 2a$ or, equivalently,

$$\max\{(n+2-2a)/(n-2), 1\} < s < (n+2)/(n-2).$$

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