A BOUND ON THE COMPLEXITY FOR $G_r T$ MODULES

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ABSTRACT. For group algebras the complexity of a module can be computed by looking at its restriction to elementary abelian subgroups. This statement is not true for modules over the restricted enveloping algebras of a restricted Lie algebra. Let G be a connected semisimple group scheme and G_r be the r th Frobenius kernel. In this paper an upper bound on the complexity is provided for $G_1 T$ modules. Furthermore, a bound is given for the complexity of a simple G_r module, $L(\lambda)$, by the complexities of the simple G_1 modules in the tensor product decomposition of $L(\lambda)$.

1. INTRODUCTION

In the study of the modular representation theory of finite groups, many global problems reduce to local problems involving elementary abelian subgroups. An example of such a result is the Alperin-Evens theorem [AE1] which states that for a finite-dimensional KG module the *complexity* of a module $[A]$, M , over KG is the maximum of the complexity of M over KE where E ranges over all elementary abelian subgroups. Now let $\mathfrak{g}$ be a restricted Lie algebra and $V(\mathfrak{g})$ be its restricted enveloping algebra. For restricted enveloping algebras the Alperin-Evens theorem is far from being true. For example, if $\mathfrak{g} = \mathfrak{sl}(2)$, the elementary abelian p -nilpotent Lie subalgebras are one-dimensional which implies that any module over these subalgebras has bounded cohomology. However, the complexity of the trivial module over $V(\mathfrak{sl}(2))$ has complexity equal to two.

In this paper we will investigate measuring the complexity of a $V(\mathfrak{g})$ module M by looking at M restricted to the root vectors of $\mathfrak{g}$. The first theorem, which can be viewed as a generalization of Cline, Parshall, and Scott's [CPS] relative injectivity theorem, provides an upper bound for the complexity of a $G_1 T$ module. For arbitrary G_r modules we will show that every finite-dimensional G_r module has finite complexity. Moreover, the complexity of a simple G_r module, $L(\lambda)$, can be shown to be bounded by the sum of the complexities of the simple G_1 modules in the Steinberg twisted tensor product decomposition of $L(\lambda)$.

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2. COMPLEXITY FOR G_r MODULES

Let G be a connected semisimple algebraic group over an algebraically closed field, K , of characteristic $p > 0$. Set G_r to be the scheme-theoretic kernel of the r th power of the Frobenius morphism $\sigma: G \rightarrow G$. It is well known that there is a categorical equivalence between rational modules for G_1 and modules for $V(\mathfrak{g})$ where $\mathfrak{g} = \text{Lie } G$. In this paper we will equip the G_r modules with an additional T structure by considering the category of $G_r T$ modules, developed by Jantzen [Jan1], which is formed by taking the pullback of T under σ on GT [CPS]. Throughout this paper we will assume that $p > h(G)$ where $h(G)$ is the Coxeter number of G .

For notational convenience we will use the conventions in [FP1]. Let M be a $V(\mathfrak{g})$ module and $H^n(V(\mathfrak{g}), M)$ denote the n th cohomology group with coefficients in M . Moreover, let the *support variety*, $|\mathfrak{g}|_M$, be the affine homogeneous variety associated with the annihilator of the commutative graded ring $H^{ev}(V(\mathfrak{g}), K)^{(-1)}$ on $H^*(V(\mathfrak{g}), M^\# \otimes M)^{(-1)}$ (the (-1) indicates that if $k \in K$, then k acts as k^p). There exists a map [Ho] from $\mathfrak{g}^\#$ (dual of $\mathfrak{g}$) into $H^2(V(\mathfrak{g}), K)^{(-1)}$ which extends to a map:

$$\Phi^*: S(\mathfrak{g}^\#) \rightarrow H^{ev}(V(\mathfrak{g}), K)^{(-1)}.$$

This map of rings induces a map on varieties $\Phi: |\mathfrak{g}|_K \rightarrow \mathbb{A}^{\dim_K \mathfrak{g}}$ whose image will be denoted by $\Phi(|\mathfrak{g}|_M) \subset \mathfrak{g}$. The image $\Phi(|\mathfrak{g}|_M)$ can be characterized as [FP1]

$$\Phi(|\mathfrak{g}|_M) = \{x \in \mathfrak{g} : x^{[p]} = 0 \text{ and } M_{(x)} \text{ is not projective}\} \cup \{0\}.$$

Finally if M is now a $G_1 T$ module, then the support variety will be T stable where the action is given by conjugation.

Let $\mathcal{V} = \{V_n : n = 0, 1, \dots\}$ be a sequence of finite-dimensional vector spaces. We say that the *rate of growth* of $\mathcal{V}$ is the smallest integer, $s \geq 0$, such that

$$\lim_{n \rightarrow \infty} \frac{\dim_K V_n}{n^s} = 0.$$

If no such nonnegative integer exists, then $\mathcal{V}$ has an *infinite* rate of growth.

For a finite-dimensional $V(\mathfrak{g})$ module, M , let P_0 denote the projective cover of M and $\Omega^1(M)$ be the kernel of $P_0 \rightarrow M$. Inductively, let P_n be the projective cover of $\Omega^{n-1}(M)$ with $\Omega^{n-1}(M)$ defined as the kernel of $P_{n-1} \rightarrow \Omega^{n-2}(M)$. By convention $\Omega^0(M) = M$. For negative integers, n , $\Omega^n(M)$ will be defined as the cokernel of $\Omega^{n+1}(M) \rightarrow Q_n$ where Q_n is the injective hull of $\Omega^{n+1}(M)$. From this construction we obtain a *minimal projective resolution* of M :

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

The complexity, $c_{\mathfrak{g}}(M)$, is the rate of growth of $\{P_n : n = 0, 1, \dots\}$. Throughout this paper we will use the fact, proved in [FP2], that

$$c_{\mathfrak{g}}(M) = \dim |\mathfrak{g}|_M = \dim \Phi(|\mathfrak{g}|_M).$$

Since G is a connected semisimple algebraic group, $\mathfrak{g} = \text{Lie } G$ will be a classical (restricted) Lie algebra. Let T be a maximal torus for G and Δ denote

the set of roots for $\mathfrak{g}$ with respect to $\mathfrak{t} = \text{Lie } T$. Moreover, since the root spaces of $\mathfrak{g}$ are one dimensional, let x_α denote a root vector in $\mathfrak{g}$ corresponding to $\alpha \in \Delta$. If $x \in \mathfrak{g}$ has the property that $x^{[p]^k} = 0$ for some $k > 0$, then $\langle x \rangle$ will denote the cycle algebra in $V(\mathfrak{g})$ generated by x . Before we state the first theorem we need to prove a proposition involving projectivity over root vectors.

Proposition 2.1. *Let x_α be a root vector corresponding to $\alpha \in \Delta$ and $L_{x_\alpha}^*$ be the kernel of the map corresponding to $\Phi(x_\alpha^*) \in H^2(V(\mathfrak{g}), K)$ (identified with $\text{Hom}_{V(\mathfrak{g})}(\Omega^2(K), K)$):*

$$0 \rightarrow L_{x_\alpha}^* \rightarrow \Omega^2(K) \rightarrow K \rightarrow 0.$$

Then $L_{x_\alpha}^|_{\langle x_\beta \rangle}$ is projective if and only if $\alpha = \beta$.*

Proof. According to [FP2, Lemma 4.1] it suffices to check whether $\Phi(x_\alpha^*)$ restricts to zero in $H^2(V(\langle x_\beta \rangle), K)$. From the correspondence between $H^2(V(\langle x_\beta \rangle), K)$ and $\text{Ext}^1(K, \langle x_\beta \rangle)$ explicitly given in [FP2, (2.1)] it suffices to check when

$$0 \rightarrow K \rightarrow K \oplus \langle x_\beta \rangle \rightarrow \langle x_\beta \rangle \rightarrow 0$$

splits as extension of restricted Lie algebras. The p th power map is defined for $a, b \in K$ to be

$$(a, b \cdot x_\beta)^{[p]} = (x_\alpha^*(b \cdot x_\beta), 0) = \begin{cases} (0, 0), & \alpha \neq \beta, \\ (1, 0), & \alpha = \beta. \end{cases}$$

Hence the extension splits precisely when $\alpha = \beta$. $\square$

We can now present the following result:

Theorem 2.2. *Let $\{x_\alpha\}$ be the set of root vectors for a classical Lie algebra $\mathfrak{g}$ relative to a maximal torus. If M is a G_1T module, then*

$$c_{\mathfrak{g}}(M) \leq \sum_{\alpha \in \Delta} c_{\langle x_\alpha \rangle}(M).$$

Proof. Let

$$\mathcal{J} = \{\alpha \in \Delta: M_{\langle x_\alpha \rangle} \text{ is not projective}\}.$$

First note that for any $x \in \mathfrak{g}$, $L_{x_\alpha}^*$ is projective as a $V(\langle x \rangle)$ module if and only if $\Omega^{-1}(L_{x_\alpha}^*)$ is projective as a $V(\langle x \rangle)$ module for $x \in \mathfrak{g}$. Therefore, by Proposition 2.1, $\Omega^{-1}(L_{x_\alpha}^*)$ is projective over $V(\langle x_\beta \rangle)$ if and only if $\alpha = \beta$.

For each $\alpha \in \mathcal{J}$ let $(X_\alpha, \delta_\alpha)$ be the exact periodic one-complex (constructed by Benson and Carlson [BC]) obtained by taking the generator $\Phi(x_\alpha^*) \in$

$H^2(V(\mathfrak{g}), K)$ [BC]:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & L_{X_\alpha^*} & \xrightarrow{\text{id}} & L_{X_\alpha^*} & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & \Omega^2(K) & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow K \longrightarrow 0 \\
 & & \downarrow \Phi(x_\alpha^*) & & \downarrow & & \downarrow \text{id} \quad \downarrow \text{id} \\
 0 & \longrightarrow & K & \longrightarrow & \Omega^{-1}(L_{X_\alpha^*}) & \longrightarrow & P_0 \longrightarrow K \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and splicing the bottom row to itself an infinite number of times. Therefore, $(X_\alpha, \delta_\alpha)$ will be given by

$$\cdots \rightarrow \Omega^{-1}(L_{X_\alpha^*}) \rightarrow P_0 \rightarrow \Omega^{-1}(L_{X_\alpha^*}) \rightarrow P_0 \rightarrow K \rightarrow 0.$$

Consider the complex $\bigotimes_{\alpha \in \mathcal{J}} (X_\alpha, \delta_\alpha) \otimes M$. From this product complex we can obtain a resolution for M through “diagonalization” [BC, Proposition 2.4]. Moreover, this resolution is a projective resolution for M since each term lying “above” and to the “left” of M in the tensor complex is projective over $V(\langle x_\alpha \rangle)$ for all $\alpha \in \Delta$ and this, projective over $V(\mathfrak{g})$ [CPS]. Hence,

$$c_{\mathfrak{g}}(M) \leq \text{card } \mathcal{J} = \sum_{\alpha \in \Delta} c_{\langle x_\alpha \rangle}(M). \quad \square$$

Remark. (1) Let $\mathfrak{g} = \mathfrak{sl}(2) = \text{span}\{x_\alpha, t, x_{-\alpha}\}$. In this case the category of G_1T modules is isomorphic to the category of graded $V(\mathfrak{g})$ modules (grading with respect to the simple roots). Consequently, the results in [N2] can be used to prove that if M is an indecomposable G_1T module, then

$$c_{\mathfrak{g}}(M) = c_{\langle x_\alpha \rangle}(M) + c_{\langle x_{-\alpha} \rangle}(M).$$

(2) For $\mathfrak{g}$ of type A_3 there exists indecomposable G_1T modules where strict inequality may hold. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ denote the simple roots and $V = K \cdot (x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_1+\alpha_2} + x_{\alpha_2+\alpha_3})$. Consider the closed irreducible conical variety $\overline{T \cdot V}$. From a computation involving the action of T on the root vectors it can be shown that $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_1+\alpha_2}, x_{\alpha_2+\alpha_3} \in \overline{T \cdot V}$. According to [FP2, Corollary 4.4] there exists a G_1T module M such that $\Phi(|\mathfrak{g}|_M) = \overline{T \cdot V}$ and without a loss of generality M can be chosen to be indecomposable since V is irreducible. However,

$$\dim \Phi(|\mathfrak{g}|_M) \leq \dim T + \dim V \leq 4 < \sum_{\alpha \in \Delta} c_{\langle x_\alpha \rangle}(M).$$

(3) Let $\mathfrak{g} = W(1, 1) = \text{span}\{e_{-1}, e_0, e_1, \dots, e_{p-2}\}$ be the p -dimensional Witt algebra. The Lie relations are given by

$$[e_i, e_j] = \begin{cases} (i-j)e_{i+j} & \text{for } -1 \leq i+j \leq p-2, \\ 0 & \text{else} \end{cases}$$

and p th power operations $e_j^{[p]} = 0$ for $j \neq 0$ and $e_0^p = e_0$. Z. Lin and the author have created a category similar to $G_1 T$ for nonclassical simple Lie algebras called $\text{Dist}(T) - V(\mathfrak{g})$ and have proved a relative injectivity result for these algebras. Using this result with a similar argument to that used in Theorem 2.2 it follows that for M a $\text{Dist}(T) - V(\mathfrak{g})$ module

$$c_{W(1,1)}(M) \leq \sum_{j \neq 0} c_{(e_j)}(M).$$

Let S be the set of simple roots of Δ and $X(T)$ be the set of characters. Moreover, let $X_r(T) = \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p^r - 1 \text{ for all } \alpha \in S\}$. Here, $\alpha^\vee$ denotes the coroot. The simple G_r modules are in one-to-one correspondence with the set $X_r(T)$. If $\lambda \in X_r(T)$ and $L(\lambda)$ is the corresponding simple G_r module, then by the Steinberg twisted tensor product theorem

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \dots \otimes L(\lambda_{r-1})^{(r-1)}$$

where $\lambda = \sum_{i=0}^{r-1} \lambda_i p^i$ with $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in X_1(T)$.

Now we will prove that the complexity of $L(\lambda)$ can be bounded by the complexity of the simple modules appearing as factors in the tensor product decomposition above. First we need to recall some elementary facts about the complexity of a module. If A is a finite-dimensional algebra over K and M is a finite-dimensional A module, then the complexity of M is equivalent to

- (1) the maximum rate of growth of $\text{Ext}_A^n(M, N)$ for all simple modules N ,
- (2) the rate of growth of $\Omega^n(M)$.

We refer the reader to [Ca3, Lemma 7.2] for a proof of the equivalence of the first statement. Note that these definitions do not use the fact that the cohomology ring is finitely generated.

Proposition 2.3. *Let $\lambda \in X_r(T)$ and $\lambda = \sum_{i=0}^{r-1} \lambda_i p^i$ with $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in X_1(T)$. Then*

$$c_{G_r}(L(\lambda)) \leq c_{G_{r-1}}(L(\lambda)) + c_{G_1^{(r-1)}}(L(\lambda_{r-1})^{(r-1)}).$$

Proof. For $\gamma \in X_r(T)$ and $\gamma = \sum_{i=0}^{r-1} \gamma_i p^i$ with $\gamma_0, \gamma_1, \dots, \gamma_{r-1} \in X_1(T)$ let $\bar{\gamma} = \sum_{i=0}^{r-2} \gamma_i p^i$. Let $L(\mu)$ for $\mu \in X_r(T)$ denote an arbitrary simple G_r module. There exists an isomorphism of group schemes [Jan3] $G_r/G_{r-1} \cong G_1^{(r-1)}$. From this isomorphism we can apply the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(G_1^{(r-1)}, H^q(G_{r-1}, L(\lambda)^\# \otimes L(\mu))) \Rightarrow H^{p+q}(G_r, L(\lambda)^\# \otimes L(\mu)).$$

Since E_∞ is a subquotient of E_2 , it suffices to find a bound for the rate of growth of E_2 . Observe that G_{r-1} acts trivially on $L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^\#^{(r-1)}$,

so there exists an isomorphism of $G_1^{(r-1)}$ modules:

$$\begin{aligned} H^q(G_{r-1}, L(\lambda)^\# \otimes L(\mu)) \\ \cong \operatorname{Ext}_{G_{r-1}}^q(L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^{\#(r-1)}, L(\bar{\lambda})^\# \otimes L(\bar{\mu})) \\ \cong L(\lambda_{r-1})^{\#(r-1)} \otimes L(\mu_{r-1})^{(r-1)} \otimes \operatorname{Ext}_{G_{r-1}}^q(L(\bar{\lambda}), L(\bar{\mu})). \end{aligned}$$

Therefore,

$$\begin{aligned} \dim_K E_2^{p,q} \\ \leq \dim_K \operatorname{Hom}_{G_1^{(r-1)}}(\Omega^p(L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^{\#(r-1)}), \operatorname{Ext}_{G_{r-1}}^q(L(\bar{\lambda}), L(\bar{\mu}))) \\ \leq (\dim_K \Omega^p(L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^{\#(r-1)}))(\dim_K \operatorname{Ext}_{G_{r-1}}^q(L(\bar{\lambda}), L(\bar{\mu}))) \\ \leq (\dim_K \Omega^p(L(\lambda_{r-1})^{(r-1)}))(\dim_K L(\mu_{r-1})^{\#(r-1)})(\dim_K \operatorname{Ext}_{G_{r-1}}^q(L(\bar{\lambda}), L(\bar{\mu}))). \end{aligned}$$

The last line is a consequence of the fact that $\Omega^p(L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^{\#(r-1)})$ is a direct summand of $\Omega^p(L(\lambda_{r-1})^{(r-1)} \otimes L(\mu_{r-1})^{\#(r-1)})$ [A]. Since G_{r-1} acts trivially on $L(\lambda_{r-1})^{(r-1)}$, it follows that $c_{G_{r-1}}(L(\lambda)) = c_{G_{r-1}}(L(\bar{\lambda}))$. According to the definitions of complexity stated above and by a similar “diagonalization” argument used in Theorem 2.2, one can deduce that the rate of growth of E_2 (i.e., $E_2^n = \sum_{p+q=n} E_2^{p,q}$) is bounded by $c_{G_1^{(r-1)}}(L(\lambda_{r-1})^{(r-1)}) + c_{G_{r-1}}(L(\bar{\lambda}))$. $\square$

From Proposition 2.3 we obtain the following theorem by using induction and the fact that $c_{G_1^{(r-1)}}(L(\lambda_{r-1})^{(r-1)}) = c_{G_1}(L(\lambda_{r-1}))$.

Theorem 2.4. *Let $\lambda \in X_r(T)$ and $\lambda = \sum_{i=0}^{r-1} \lambda_i p^i$ with $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in X_1(T)$. Then*

$$c_{G_r}(L(\lambda)) \leq \sum_{i=0}^{r-1} c_{G_1}(L(\lambda_i)).$$

For $\lambda = 0$ we can conclude that $c_{G_r}(K) \leq r \cdot c_{G_1}(K)$. Since the distribution algebra, $\operatorname{Dist}(G_r)$, is a finite-dimensional Hopf algebra, $c_{G_r}(M) \leq c_{G_r}(K)$ for M a finite-dimensional G_r module. Consequently, every finite-dimensional G_r module has finite complexity. This statement would follow immediately from finite generation of cohomology (i.e., $H^*(G_r, K)$ is Noetherian), but this is still an open problem for $r > 2$ [FP2, (5.4)].

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