THE CASSELMAN-SHALIKA FORMULA FOR A DISTINGUISHED MODEL

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ABSTRACT. Unramified Whittaker functions and their analogues occur naturally in number theory as terms in the Fourier expansions of automorphic forms. Precise information about these functions is useful in many aspects of study, such as in the construction of L-functions. In this paper, the method of Casselman and Shalika is used to derive explicit values for the analogue of the unramified Whittaker function in a distinguished model that arises in connection with quadratic base change.

1. INTRODUCTION

The unramified Whittaker functions and their analogues play an important role in modern number theory, arising naturally as terms in the Fourier expansions of automorphic forms. It is generally desirable (whenever possible) to calculate explicit values for these functions, as the information proves useful in many aspects of study related to the automorphic form (for example, in the construction of associated L-functions). When an automorphic representation possesses a Whittaker model or another suitable unique model, the method described in [3] may be used to compute an explicit formula (the *Casselman-Shalika formula*) for the values of the unramified Whittaker function (or the analogous function).

In this paper, we consider the following *distinguished model* that arises in connection with quadratic base change and the theory of the Asai L-function (see [1], [5], and [6]). The existence and uniqueness of this model was established by Hakim in [4]. Let F be a nonarchimedean local field of characteristic zero and with odd residue characteristic, and let q_F denote the cardinality of the residue field. Let E/F be an unramified quadratic extension of F . Let $\mathcal{O}_E$ denote the ring of integers in E . Put $G_E = GL(2, E)$, $G_F = GL(2, F)$, and $K_E = GL(2, \mathcal{O}_E)$. If (π, V) is an irreducible, admissible, unramified principal series representation of G_E that is trivial on the center Z_E of G_E , it is known that:

- (i) There exists a nonzero spherical vector $\phi^0 \in V$, unique up to complex constant, such that $\pi(k)\phi^0 = \phi^0$ for all $k \in K_E$.

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- (ii) There exists a nonzero linear functional $T : V \rightarrow \mathbb{C}$, unique up to complex constant, such that $T(\pi(g)\phi) = T(\phi)$ for all $g \in G_F$ and $\phi \in V$ (see [4]).

As π is in the principal series, ϕ^0 may be regarded as a complex-valued function on G_E . In §6, it is shown that $T(\phi^0) \neq 0$. Thus, we can normalize the constants in (i) and (ii) so that $\phi^0 = 1$ on K_E , and $T(\phi^0) = 1$. Now let $\mathcal{Q}$ be the vector space of all locally-constant complex-valued functions on G_E that satisfy $f(gg') = f(g')$ for all $g \in G_F$ and $g' \in G_E$. The group G_E acts on $\mathcal{Q}$ by right translation. Let $\mathcal{Q}(\pi)$ denote the invariant subspace of $\mathcal{Q}$ spanned by functions Q_ϕ of the form $Q_\phi(g) = T(\pi(g)\phi)$. The map $\phi \mapsto Q_\phi$ yields an isomorphism of G_E -modules $V \cong \mathcal{Q}(\pi)$, and $\mathcal{Q}(\pi)$ is called the *distinguished model* for V .

For the above model, the correct analogue of the unramified Whittaker function is the function $Q_0 = Q_{\phi^0}$ given by $Q_0(g) = T(\pi(g)\phi^0)$ for all $g \in G_E$. Observe that Q_0 is a well-defined function on the double cosets of $G_F \backslash G_E / K_E$, hence it suffices to compute Q_0 on the complete set of double coset representatives $\{M_k : k = 0, 1, 2, \dots\}$ described in §3. The main result of this paper (proved in §6) is the following:

Theorem. Let $t_\chi \in GL(2, \mathbb{C})$ be the Satake parameter of π , and let $\vee^k$ be the k th symmetric tensor representation of $GL(2, \mathbb{C})$ for each $k \geq 0$. Then:

$$Q_0(M_0) = 1,$$

$$Q_0(M_1) = q_F^{-1}(q_F - 1)^{-1} \left(q_F \operatorname{Tr}(t_\chi) - (q_F + 1) \right),$$

and if $k \geq 2$

$$Q_0(M_k) = q_F^{-k}(q_F - 1)^{-1} \left(q_F \operatorname{Tr}(\vee^k t_\chi) - (q_F + 1) \operatorname{Tr}(\vee^{k-1} t_\chi) + \operatorname{Tr}(\vee^{k-2} t_\chi) \right).$$

2. SOME NOTATION

Let v be a fixed valuation of E (which restricts to a valuation of F), and fix once and for all a prime element ϖ in F such that $v(\varpi) = 1$ (this is possible since E/F is unramified).

Let $\mathcal{O}_E$ denote the ring of integers of E , $\wp_E$ its unique maximal ideal, and $\mathcal{O}_E^\times$ its group of units. Let q_E be the (odd) cardinality of the residue field $\mathcal{O}_E/\wp_E$, and let $|\cdot|_E = q_E^{-v(\cdot)}$ be the absolute value symbol for E corresponding to v . We similarly define $\mathcal{O}_F$, $\wp_F$, $\mathcal{O}_F^\times$, q_F , and $|\cdot|_F = q_F^{-v(\cdot)}$. Then $q_E = q_F^2$ and $|\cdot|_E = |\cdot|_F^2$.

As E/F is quadratic, $E = F(\mu)$ for some $\mu \in \mathcal{O}_E^\times - \mathcal{O}_F^\times$ with $\mu^2 \in \mathcal{O}_F^\times$. Then $E = F \oplus \mu F$ and $\mathcal{O}_E = \mathcal{O}_F \oplus \mu \mathcal{O}_F$. By Hensel's lemma, the image of $\tau = \mu^2$ in the residue field $\mathcal{O}_F/\wp_F$ is a quadratic nonresidue.

For $G = GL(2)$, let P be the standard Borel subgroup of G , let N be the unipotent radical of P , and let Z be the center of G . The Weyl group W of G consists of the elements $\{e, w_0\}$, where $e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $w_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Once and for all, fix $\xi = \begin{pmatrix} 1 & \mu \\ & 1 \end{pmatrix} \in G_E$.

An additive Haar measure dx on F is said to be *normalized* if $\operatorname{vol}(\mathcal{O}_F) = 1$. A multiplicative Haar measure $d^\times y$ on $F^\times$ is said to be *normalized* if $d^\times y = |y|_F^{-1} dy$ for some normalized additive Haar measure dy on F . By the

Bruhat decomposition, $G_F = P_F \cup P_F w_0 P_F$. Then $Z_F \backslash P_F$ has measure 0 in $Z_F \backslash G_F$ and the matrices

$$\left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} : x, \bar{x} \in F \text{ and } y \in F^\times \right\}$$

form a complete set of distinct representatives for $Z_F \backslash P_F w_0 P_F$. A left Haar measure dg is said to be *normalized* if

$$\int_{Z_F \backslash G_F} \phi(g) dg = \int_F \int_{F^\times} \int_F \phi \left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} d\bar{x}$$

for all ϕ integrable on $Z_F \backslash G_F$, where dx and $d\bar{x}$ are normalized additive Haar measures for F , and $d^\times y$ is a normalized multiplicative Haar measure for $F^\times$.

3. DOUBLE COSET DECOMPOSITION

The function Q_0 described in §1 is well defined on $G_F \backslash G_E / K_E$, since

$$Q_0(gg'k) = T(\pi(gg'k)\phi^0) = T(\pi(g)\pi(g')\pi(k)\phi^0) = T(\pi(g')\phi^0) = Q_0(g')$$

for all $g \in G_F$, $g' \in G_E$, and $k \in K_E$. Hence, it suffices to determine Q_0 on a complete set of double coset representatives for $G_F \backslash G_E / K_E$.

By the Iwasawa decomposition, a complete set of double coset representatives for $N_E \backslash G_E / K_E$ is given by the matrices $\mathcal{A} = \left\{ \begin{pmatrix} \varpi^k & \\ & \varpi^l \end{pmatrix} : k, l \in \mathbb{Z} \right\}$. Then $N_E \mathcal{A}$ is a complete set of representatives for G_E / K_E . As $N_E \mathcal{A} = \mathcal{A} N_E$ and $\mathcal{A} \subset G_F$, it follows that N_E is a complete set of representatives for $G_F \backslash G_E / K_E$. To reduce the set further, note that each $z \in E$ may be written in the form $z = x + u\varpi^{-k}\mu$, where $x \in F$, $u \in \mathcal{O}_F^\times$, and $k \in \mathbb{Z}$. Then

$$\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} = \begin{pmatrix} u\varpi^{-k} & x \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^k & \mu \\ & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix},$$

with $\begin{pmatrix} u\varpi^{-k} & x \\ & 1 \end{pmatrix} \in G_F$ and $\begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} \in K_E$. Moreover, since $\begin{pmatrix} \varpi^k & \mu \\ & 1 \end{pmatrix} \in G_F K_E$ whenever $k \leq 0$, it follows that the matrices $\left\{ M_k = \begin{pmatrix} \varpi^k & \mu \\ & 1 \end{pmatrix} : k \in \mathbb{Z}, k \geq 0 \right\}$ form a complete set of double coset representatives for $G_F \backslash G_E / K_E$. Moreover, it may be shown that these matrices represent *distinct* cosets.

4. THE LINEAR FUNCTIONAL T

As a principal series representation, the space $V = V_\chi$ of π is the vector space $\text{Ind}_{P_E}^{G_E}(\delta^{\frac{1}{2}}\chi)$ consisting of all locally-constant complex-valued functions on G_E that satisfy $\phi(pg') = (\delta^{\frac{1}{2}}\chi)(p)\phi(g')$ for all $p \in P_E$ and $g' \in G_E$. Here χ and δ are characters of P_E :

$$\begin{aligned} \chi \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} &= \chi_1(y_1)\chi_2(y_2), \\ \delta \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} &= \left| \frac{y_1}{y_2} \right|_E = q_E^{v(y_2) - v(y_1)} \end{aligned}$$

for all $y_1, y_2 \in E^\times$ and $* \in E$, where χ_1, χ_2 are characters of $E^\times$. Since π is unramified, χ is unramified, and we can assume that

$$\chi_1(y) = \alpha^{v(y)}, \quad \chi_2(y) = \beta^{v(y)}$$

for some $\alpha, \beta \in \mathbb{C}^\times$ and all $y \in E^\times$. Moreover, the condition that π is trivial on Z_E implies $\alpha\beta = 1$. Thus,

$$\phi \left(\begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} g \right) = (\beta q_F)^{v(y_2) - v(y_1)} \phi(g)$$

for all $y_1, y_2 \in E^\times$, $* \in E$, $g \in G$, and $\phi \in V_\chi$.

An unramified character χ of P_E as above is said to *dominant* if $|\beta| > 1$.

Proposition 1. *Suppose that χ is a dominant character and dg is a normalized left Haar measure on $Z_F \backslash G_F$. Then the integral*

$$\tilde{T}(\phi) = \int_{Z_F \backslash G_F} \phi(w_0 \xi g) dg$$

is well defined and absolutely convergent for all $\phi \in V_\chi$, and therefore defines a linear functional $\tilde{T} : V_\chi \rightarrow \mathbb{C}$ such that

$$\tilde{T}(\pi(g)\phi) = \tilde{T}(\phi)$$

for all $g \in G_F$ and $\phi \in V_\chi$.

Proof. The only issue here is that of convergence. By uniqueness of Haar measure, the integral $\tilde{T}(\phi)$ is a nonzero multiple of the integral:

$$\int_{K_F} \int_{F^\times} \int_F \phi \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k \right) \frac{dx d^\times y}{|y|_F} dk,$$

where dk is a left Haar measure on $K_F = GL(2, \mathcal{O}_F)$, dx is a normalized additive Haar measure for F , and $d^\times y$ is a normalized multiplicative Haar measure for $F^\times$.

For each $k \in K_F$ and $\phi \in V_\chi$, let

$$\mathcal{G}_\phi(k) = \int_{F^\times} \int_F \phi \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k \right) \frac{dx d^\times y}{|y|_F}.$$

Then each $\mathcal{G}_\phi$ is locally-constant on K_F (since ϕ is locally-constant). As K_F is compact, the absolute convergence of $\tilde{T}(\phi)$ follows from the absolute convergence of the integrals $\mathcal{G}_\phi(k)$. Moreover, there is no loss of generality in

assuming that $k = e$, since $\mathcal{G}_\phi(k) = \mathcal{G}_{\pi(k)\phi}(e)$. Thus,

$$\begin{aligned}
 \mathcal{G}_\phi(e) &= \int_{F^\times} \int_F \phi \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \right) \frac{dx \, d^\times y}{|y|_F} \\
 &= \int_{F^\times} \int_F \phi \left(w_0 \xi \begin{pmatrix} y^{-1} & x \\ & 1 \end{pmatrix} \right) |y|_F \, dx \, d^\times y \\
 &= \int_{F-\{0\}} \int_F \phi \left(w_0 \xi \begin{pmatrix} y^{-1} & x \\ & 1 \end{pmatrix} \right) dx \, dy \\
 &= \int_{F-\{0\}} \int_F (\beta q_F)^{-v(y)} \phi \left(w_0 \begin{pmatrix} 1 & xy + y\mu \\ & 1 \end{pmatrix} \right) dx \, dy \\
 &= \int_{F-\{0\}} \int_F \beta^{-v(y)} \phi \left(w_0 \begin{pmatrix} 1 & x + y\mu \\ & 1 \end{pmatrix} \right) dx \, dy \\
 &= \int_F \int_{F-\{0\}} \beta^{-v(y)} \phi \left(w_0 \begin{pmatrix} 1 & x + y\mu \\ & 1 \end{pmatrix} \right) dy \, dx
 \end{aligned}$$

by Fubini's theorem. Since χ is dominant ($|\beta| > 1$), we can add the point $y = 0$ to each inner integral to obtain

$$\begin{aligned}
 \mathcal{G}_\phi(e) &= \int_F \int_F \beta^{-v(y)} \phi \left(w_0 \begin{pmatrix} 1 & x + y\mu \\ & 1 \end{pmatrix} \right) dy \, dx \\
 &= \int_E \beta^{-v(\operatorname{Im}(z))} \phi \left(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \right) dz,
 \end{aligned}$$

where $\operatorname{Im}(z) = y$ if $z = x + y\mu$, and $dz = dy \, dx$ is a normalized additive Haar measure for E . Since ϕ is locally constant, we can choose $L \gg 0$ so that $\phi(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}) = \phi(w_0)$ if $v(z) \geq L$, and $\phi \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} = \phi(e)$ if $v(z) \leq -L$. Also whenever $z \neq 0$ we have

$$\begin{aligned}
 \phi \left(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \right) &= \phi \left(\begin{pmatrix} 1 & z^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -z^{-1} & \\ & z \end{pmatrix} \begin{pmatrix} 1 & \\ z^{-1} & 1 \end{pmatrix} \right) \\
 &= (\beta q_F)^{2v(z)} \phi \left(\begin{pmatrix} 1 & \\ z^{-1} & 1 \end{pmatrix} \right).
 \end{aligned}$$

It follows that $\mathcal{G}_\phi(e)$ can be expressed as a sum of three integrals:

$$\begin{aligned}
 &\phi(w_0) \int_{\substack{E \\ v(z) \geq L}} \beta^{-v(\operatorname{Im}(z))} dz + \phi(e) \int_{\substack{E \\ v(z) \leq -L}} \beta^{-v(\operatorname{Im}(z))} (\beta q_F)^{2v(z)} dz \\
 &+ \int_{\substack{E \\ -L < v(z) < L}} \beta^{-v(\operatorname{Im}(z))} \phi \left(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \right) dz.
 \end{aligned}$$

The first two integrals converge absolutely (by a straightforward calculation) whenever $|\beta| > 1$, and the last integral converges absolutely for all β since it is the integral of a continuous function over a compact region. $\square$

For dominant characters χ , the lemma in §6 shows that $\tilde{T}(\phi^0) \neq 0$, and we can define

$$T(\phi) = \frac{\tilde{T}(\phi)}{\tilde{T}(\phi^0)}$$

for $\phi \in V_\chi$. Then $T : V_\chi \rightarrow \mathbb{C}$ is the unique linear functional that satisfies $T(\pi(g)\phi) = T(\phi)$ for all $g \in G_F$, $\phi \in V_\chi$, and $T(\phi^0) = 1$.

A *flat section* is a collection $\{\phi_\chi \in \text{Ind}_{P_E}^{G_E}(\delta^{\frac{1}{2}}\chi) : \chi \text{ unramified}\}$ such that $\phi_\chi|_{K_E}$ is independent of χ . It follows from the Iwasawa decomposition that every flat section $\{\phi_\chi\}$ is *equicontinuous*; that is, there exists an open neighborhood K_ϕ of the identity, which is independent of χ , such that ϕ_χ is K_ϕ -fixed for every χ .

Proposition 2. *Let g' be a fixed element of G_E , and let $\{\phi_\chi\}$ be a flat section. Then the function $(1 - (\beta q_F)^{-2})T(\pi(g')\phi_\chi)$, initially defined for dominant characters χ (i.e., $|\beta| > 1$), extends to an entire function of $\beta \in \mathbb{C}$.*

Proof. From the proof of Proposition 1 (when χ is dominant)

$$\begin{aligned} \mathcal{E}_{\pi(g')\phi_\chi}(k) = & \phi_\chi(w_0 k g') \int_{\substack{E \\ v(z) \geq L_\chi(k)}} \beta^{-v(\text{Im}(z))} dz \\ & + \phi_\chi(k g') \int_{\substack{E \\ v(z) \leq -L_\chi(k)}} \beta^{-v(\text{Im}(z))} (\beta q_F)^{2v(z)} dz \\ & + \int_{\substack{E \\ -L_\chi(k) < v(z) < L_\chi(k)}} \beta^{-v(\text{Im}(z))} \phi_\chi \left(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} k g' \right) dz, \end{aligned} \quad (1)$$

for each unramified character χ and $k \in K_F$, where $L_\chi(k) \gg 0$ is chosen so that

$$\begin{aligned} v(z) \geq L_\chi(k) &\implies \phi_\chi \left(w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} k g' \right) = \phi_\chi(w_0 k g'), \\ v(z) \leq -L_\chi(k) &\implies \phi_\chi \left(\begin{pmatrix} 1 & & \\ & z^{-1} & \\ & & 1 \end{pmatrix} k g' \right) = \phi_\chi(k g'). \end{aligned}$$

As $\{\phi_\chi\}$ is equicontinuous, we can choose $L_\chi(k) = L(k)$ independent of χ and locally-constant on K_F . As K_F is compact, we can choose $L(k) = L$ independent of k .

The first two integrals in (1) may be directly computed for dominant χ :

$$\phi(w_0 k g') \int_{v(z) \geq L} \beta^{-v(\text{Im}(z))} dz = \phi(w_0 k g') \frac{(\beta q_F^2)^{-L}}{1 - (\beta q_F)^{-1}}, \quad (2)$$

$$\begin{aligned} \phi_\chi(k g') \int_{v(z) \leq -L} \beta^{-v(\text{Im}(z))} (\beta q_F)^{2v(z)} dz \\ = \phi_\chi(k g') \frac{\beta^{-L}(1 - q_F^{-1})(1 - (\beta q_F^2)^{-1})}{(1 - (\beta q_F)^{-1})(1 - \beta^{-1})}. \end{aligned} \quad (3)$$

As before, the last integral in (1) converges absolutely for all unramified characters χ and all $k \in K_F$ and consequently defines an *entire* function of β .

It follows that for each $k \in K_F$, $(1 - (\beta q_F)^{-1})(1 - \beta^{-1})\mathcal{G}_{\pi(g')\phi_\chi}(k)$ continues to an entire function of β , and as K_F is compact, it follows that $(1 - (\beta q_F)^{-1})(1 - \beta^{-1})\tilde{T}(\pi(g')\phi_\chi)$ has this continuation as well. For each unramified character χ , let ϕ_χ^0 be the unique K_E -fixed vector in $\text{Ind}_{P_E}^{G_E}(\delta^{\frac{1}{2}}\chi)$ such that $\phi_\chi^0 = 1$ on K_E . Then the collection $\{\phi_\chi^0\}$ is a flat section. The lemma in §6 shows that

$$\tilde{T}(\phi_\chi^0) = \frac{(1 - q_F^{-2})(1 + (\beta q_F)^{-1})}{(1 - \beta^{-1})}$$

for dominant χ , hence:

$$(1 - (\beta q_F)^{-2})T(\pi(g')\phi_\chi) = \frac{(1 - (\beta q_F)^{-1})(1 - \beta^{-1})\tilde{T}(\pi(g')\phi_\chi)}{(1 - q_F^{-2})}.$$

This equation implies the proposition. $\square$

5. IWAHORI FIXED VECTORS

Let B_E be the *Iwahori subgroup* of G_E (i.e., the inverse image of the standard Borel subgroup under the natural homomorphism $K_E \rightarrow GL(2, \mathcal{O}_E/\varphi_E)$). Let $V_\chi^{B_E}$ be the space of *Iwahori fixed vectors*:

$$V_\chi^{B_E} = \{\phi \in V_\chi : \pi(b)\phi = \phi \text{ for all } b \in B_E\}.$$

By the Iwahori factorization, each element $g' \in G_E$ may be expressed in the form $g' = p\bar{w}_{g'}b$ where $p \in P_E$, $\bar{w}_{g'} \in W$, and $b \in B_E$. Moreover, $\bar{w}_{g'}$ is uniquely determined by g' . For each $w \in W$, let

$$\phi_{w,\chi}(g') = \begin{cases} (\delta^{\frac{1}{2}}\chi)(p) & \text{if } g' = p\bar{w}_{g'}b \text{ and } \bar{w}_{g'} = w, \\ 0 & \text{otherwise.} \end{cases}$$

By a theorem of Casselman (see [2]), the vectors $\{\phi_{w,\chi}\}_{w \in W}$ form a basis (the *Casselman basis*) for $V_\chi^{B_E}$.

Let $N(\mathcal{O}_E) = N_E \cap K_E$, and let dn be a left Haar measure on N_E such that $\text{vol}(N(\mathcal{O}_E)) = 1$. For each $k \geq 0$ and $g' \in G_E$, the integral

$$F_k(g') = \int_{N(\mathcal{O}_E)} \phi_\chi^0 \left(g'n \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} \right) dn$$

converges absolutely for all unramified χ . Note that $F_0 = \phi_\chi^0$. It is known that the $\{F_k\}$ are Iwahori fixed vectors given explicitly by

$$(4) \quad F_k = \sum_{w \in W} c_w(\chi)(\delta^{\frac{1}{2}}w\chi) \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} \phi_{w,\chi}$$

where $c_e(\chi) = 1$, $c_{w_0}(\chi) = (1 - (\beta q_F)^{-2})(1 - \beta^{-2})^{-1}$, and ${}^w\chi$ is the character given by

$${}^w\chi \begin{pmatrix} y_1 & * \\ & y_2 \end{pmatrix} = \chi \left(w^{-1} \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} w \right)$$

for all $y_1, y_2 \in E^\times$ and $* \in E$.

Proposition 3. For dominant χ , $Q_0(M_0) = T(F_0)$, and if $k \geq 1$,

$$Q_0(M_k) = (1 - q_F^{-1})^{-1} T(F_k - q_F^{-1} F_{k-1}).$$

Proof. We compute:

$$\begin{aligned}
 \tilde{T}(F_k) &= \int_{Z_F \backslash G_F} F_k(w_0 \xi g) dg \\
 &= \int_{Z_F \backslash G_F} \int_{N(\mathcal{O}_E)} \phi_\chi^0 \left(w_0 \xi g n \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} \right) dn dg \\
 (5) \quad &= \int_{Z_F \backslash G_F} \int_{\mathcal{O}_F} \int_{\mathcal{O}_F} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x\mu \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} \right) dy dx dg \\
 &= \int_{Z_F \backslash G_F} \int_{\mathcal{O}_F} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} \varpi^k & x\mu \\ & 1 \end{pmatrix} \right) dx dg
 \end{aligned}$$

since $\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \in G_F$ for all $y \in \mathcal{O}_F$ and $\text{vol}(\mathcal{O}_F) = 1$.

If $x = u\varpi^n$ with $u \in \mathcal{O}_F^\times$, $n \in \mathbb{Z}$, then $\begin{pmatrix} u\varpi^n & \\ & 1 \end{pmatrix} \in G_F$, $\begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} \in K_E$, and therefore,

$$\begin{aligned}
 &\int_{Z_F \backslash G_F} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} \varpi^k & x\mu \\ & 1 \end{pmatrix} \right) dg \\
 &= \int_{Z_F \backslash G_F} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} u\varpi^n & \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^{k-n} & \mu \\ & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} \right) dg \\
 &= \int_{Z_F \backslash G_F} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} \varpi^{k-n} & \mu \\ & 1 \end{pmatrix} \right) dg \\
 &= \tilde{T} \left(\pi \begin{pmatrix} \varpi^{k-n} & \mu \\ & 1 \end{pmatrix} \phi_\chi^0 \right) = \begin{cases} \tilde{T}(\phi_\chi^0) Q_0(M_0) & \text{if } n \geq k, \\ \tilde{T}(\phi_\chi^0) Q_0(M_{k-n}) & \text{if } n < k. \end{cases}
 \end{aligned}$$

Thus, applying Fubini's theorem to (5), it follows that

$$\begin{aligned}
 \tilde{T}(F_k) &= \sum_{n \geq 0} \int_{v(x)=n} \phi_\chi^0 \left(w_0 \xi g \begin{pmatrix} \varpi^{k-n} & x\mu \\ & 1 \end{pmatrix} \right) dg dx \\
 &= \sum_{n \geq k} \text{vol}(\wp_F^n - \wp_F^{n+1}) \tilde{T}(\phi_\chi^0) Q_0(M_0) \\
 &\quad + \sum_{n=0}^{k-1} \text{vol}(\wp_F^n - \wp_F^{n+1}) \tilde{T}(\phi_\chi^0) Q_0(M_{k-n}) \\
 &= q_F^{-k} \tilde{T}(\phi_\chi^0) Q_0(M_0) + (1 - q_F^{-1}) \tilde{T}(\phi_\chi^0) \sum_{n=0}^{k-1} q_F^{-n} Q_0(M_{k-n}).
 \end{aligned}$$

The proposition now follows from this by a simple inductive argument. $\square$

6. CALCULATION OF $T(\phi_w, \chi)$ AND $Q_0(M_k)$

Lemma. For dominant χ , we have

$$\tilde{T}(\phi_\chi^0) = \frac{(1 - q_F^{-2})(1 + (\beta q_F)^{-1})}{(1 - \beta^{-1})}.$$

Proof. The left Haar measure dg is normalized (see §2) so that:

$$\begin{aligned} \tilde{T}(\phi_\chi^0) &= \int_{Z_F \backslash G_F} \phi_\chi^0(w_0 \xi g) dg \\ &= \int_F \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} d\bar{x}. \end{aligned}$$

If $\bar{x} \in \mathcal{O}_F$, then $w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \in K_F$ and

$$\begin{aligned} &\int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} \\ &= \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F}. \end{aligned}$$

On the other hand, if $\bar{x} \notin \mathcal{O}_F$, then $\begin{pmatrix} 1 & \\ & \bar{x}^{-1} \end{pmatrix} \in K_F$ and

$$\begin{aligned} &\int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} \\ &= \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{x}^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -\bar{x}^{-1} & \\ & \bar{x} \end{pmatrix} \begin{pmatrix} 1 & \\ & \bar{x}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} \\ &= \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} -y\bar{x}^{-2} & x + y\bar{x}^{-1} \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} \\ &= |\bar{x}|_F^{-2} \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F}. \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{T}(\phi_\chi^0) &= \int_F q_F^{\min(2v(x), 0)} d\bar{x} \int_{F^\times} \int_F \phi_\chi^0 \left(w_0 \xi \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \right) \frac{dx d^\times y}{|y|_F} \\ &= (1 + q_F^{-1}) \mathcal{G}_{\phi_\chi^0}(e). \end{aligned}$$

Since ϕ_χ^0 is K_E -fixed, it follows from equation (1) of §4 that

$$\mathcal{G}_{\phi_\chi^0}(e) = \int_E \beta^{-v(\operatorname{Im}(z))} (\beta q_F)^{2v(z)} dz + \int_E \beta^{-v(\operatorname{Im}(z))} dz.$$

$v(z) \leq -1$ $v(z) \geq 0$

To evaluate these integrals, take $L = 0$, $\phi = \phi_\chi^0$ in equation (2) of §4, and $L = 1$, $\phi = \phi_\chi^0$ in equation (3) of §4. The lemma follows. $\square$

We now turn to the calculation of $\tilde{T}(\phi_{w,\chi})$ for $w \in W$. For this, it will be necessary to determine those $g' = w_0 \xi \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ 1 & 1 \end{pmatrix}$ in G such that $\bar{w}_{g'} = w_0$. Thus, let $\lambda = y^{-1}(x + \mu)$ and $\eta = \lambda^{-1} + \bar{x} = y(x - \mu)(x^2 - \tau)^{-1} + \bar{x}$, where $\tau = \mu^2$. Then

$$\begin{aligned} w_0 \xi \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & \\ & y \end{pmatrix} \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & \bar{x} \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & y \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} -\lambda^{-1} & \\ & \lambda \end{pmatrix} w_0 \begin{pmatrix} 1 & \eta \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

It follows that $\bar{w}_{g'} = w_0$ if and only if $\eta \in \mathcal{O}_E$ and hence, if and only if $y(x^2 - \tau)^{-1} \in \mathcal{O}_F$ and $\bar{x} + yx(x^2 - \tau)^{-1} \in \mathcal{O}_F$. Consequently (if $\bar{w}_{g'} = w_0$),

$$\phi_{w_0,\chi}(g') = (\beta q_F)^{v(y\lambda^2)} = (\beta q_F)^{(\min(2v(x), 0) - v(y))}.$$

Proposition 4. For dominant χ , we have

$$T(\phi_{e,\chi}) = \frac{(\beta q_F)^{-1} - \beta^{-2}}{1 - \beta^{-2}} \quad \text{and} \quad T(\phi_{w_0,\chi}) = \frac{1}{1 + (\beta q_F)^{-1}}.$$

Proof. In this proof, Fubini's theorem will be applied frequently without mention.

We compute

$$\begin{aligned} \tilde{T}(\phi_{w_0,\chi}) &= \int_F \int_{F^\times} \int_F \phi_{w_0,\chi} \left(w_0 \xi \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} w_0 \begin{pmatrix} 1 & \bar{x} \\ 1 & 1 \end{pmatrix} \right) \frac{dx \, d^\times y}{|y|_F} \, d\bar{x} \\ &= \int_F \int_F \int_{F^\times} (\beta q_F)^{(\min(2v(x), 0) - v(y))} \frac{d^\times y \, dx}{|y|_F} \, d\bar{x} \\ &\quad y(x - \mu)(x^2 - \tau)^{-1} + \bar{x} \in \mathcal{O}_E \end{aligned}$$

We apply a change of variables $y \mapsto (x^2 - \tau)y_o$, $d^\times y \mapsto d^\times y_o$, where $y_o \in \mathcal{O}_F - \{0\}$. Because the image of τ in $\mathcal{O}_F/\mathfrak{o}_F$ is a quadratic nonresidue (see §2), it follows that $v(y) = \min(2v(x), 0) + v(y_o)$, and thus

$$\tilde{T}(\phi_{w_0,\chi}) = \int_F \int_F \int_{\mathcal{O}_F - \{0\}} (\beta q_F)^{-v(y_o)} \frac{1}{|x^2 - \tau|_F} \frac{1}{|y_o|_F} d^\times y_o \, dx \, d\bar{x}.$$

Next, we apply a change of variables $x \mapsto y_o^{-1}(x_o - \bar{x})$, $dx \mapsto |y_o|_F dx_o$, where $x_o \in \mathcal{O}_F$:

$$\begin{aligned} \tilde{T}(\phi_{w_0,\chi}) &= \int_F \int_{\mathcal{O}_F - \{0\}} \int_{\mathcal{O}_F} (\beta q_F)^{-v(y_o)} \frac{1}{\left| (y_o^{-1}(x_o - \bar{x}))^2 - \tau \right|_F} \frac{1}{|y_o|_F^2} dx_o \, d^\times y_o \, d\bar{x} \\ &= \int_{\mathcal{O}_F - \{0\}} \int_{\mathcal{O}_F} \beta^{-v(y_o)} q_F^{v(y_o)} \left(\int_F \frac{d\bar{x}}{\left| (y_o^{-1}(x_o - \bar{x}))^2 - \tau \right|_F} \right) dx_o \, d^\times y_o. \end{aligned}$$

To compute the inner integral, we apply a change of variables $\bar{x} \mapsto y_o \bar{x}_o + x_o$,

$d\bar{x} \mapsto |y_o|_F d\bar{x}_o :$

$$\begin{aligned} \int_F \frac{d\bar{x}}{|(y_o^{-1}(x_o - \bar{x}))^2 - \tau|_F} &= |y_o|_F \int_F \frac{d\bar{x}_o}{|\bar{x}_o^2 - \tau|_F} \\ &= |y_o|_F \int_F q_F^{\min(2v(\bar{x}_o), 0)} d\bar{x}_o = |y_o|_F (1 + q_F^{-1}). \end{aligned}$$

Then

$$\tilde{T}(\phi_{w_0, \chi}) = (1 + q_F^{-1}) \int_{\mathcal{O}_F - \{0\}} \int_{\mathcal{O}_F} \beta^{-v(y_o)} dx_o d^\times y_o = \frac{(1 - q_F^{-2})}{(1 - \beta^{-1})},$$

and therefore

$$T(\phi_{w_0, \chi}) = \frac{\tilde{T}(\phi_{w_0, \chi})}{\tilde{T}(\phi_\chi^0)} = \frac{1}{1 + (\beta q_F)^{-1}}.$$

From §4, we know that $F_0 = \phi_\chi^0$; thus, using equation (4) we can write

$$\phi_\chi^0 = c_e(\chi)\phi_{e, \chi} + c_{w_0}(\chi)\phi_{w_0, \chi}.$$

Applying the linear functional T , we obtain

$$\begin{aligned} 1 &= c_e(\chi)T(\phi_{e, \chi}) + c_{w_0}(\chi)T(\phi_{w_0, \chi}) \\ &= T(\phi_{e, \chi}) + \frac{1 + (\beta q_F)^{-1}}{1 - \beta^{-2}}, \end{aligned}$$

whence the proposition follows. $\square$

Proof of Theorem. By Proposition 2, each $Q_0(M_k)$ (regarded as a function of β) continues meromorphically to the entire complex plane, with possible (simple) poles at $\pm q_F^{-1}$.

Applying the linear functional T to equation (4), we have by Proposition 4

$$\begin{aligned} T(F_k) &= \sum_{w \in W} c_w(\chi)(\delta^{\frac{1}{2}w}\chi) \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} T(\phi_w, \chi) \\ &= q_F^{-k} \beta^{-k} \frac{(\beta q_F)^{-1} - \beta^{-2}}{1 - \beta^{-2}} + \frac{1 - (\beta q_F)^{-2}}{1 - \beta^{-2}} q_F^{-k} \beta^k \frac{1}{1 + (\beta q_F)^{-1}}. \end{aligned}$$

Substituting $\alpha = \beta^{-1}$, this simplifies to

$$\begin{aligned} T(F_k) &= q_F^{-k} \left(\alpha^k \frac{\alpha - q_F^{-1}}{\alpha - \beta} - \beta^k \frac{\beta - q_F^{-1}}{\alpha - \beta} \right) \\ &= q_F^{-k} \left(\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} - q_F^{-1} \frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \\ &= q_F^{-k} (S_k(\alpha, \beta) - q_F^{-1} S_{k-1}(\alpha, \beta)), \end{aligned}$$

where $S_k(\alpha, \beta)$ is defined for $k \in \mathbb{Z}$ by

$$S_k(\alpha, \beta) = \begin{cases} \sum_{i=0}^k \alpha^i \beta^{k-i} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each $S_k(\alpha, \beta)$ is an *entire* function of β .

Now, by Proposition 3 (when χ is dominant), $Q_0(M_0) = T(F_0) = 1$, and when $k \geq 1$:

$$\begin{aligned} Q_0(M_k) &= (1 - q_F^{-1})^{-1} T(F_k - q_F^{-1} F_{k-1}) = (1 - q_F^{-1})^{-1} (T(F_k) - q_F^{-1} T(F_{k-1})) \\ &= q_F^{-k} (q_F - 1)^{-1} (q_F S_k(\alpha, \beta) - (q_F + 1) S_{k-1}(\alpha, \beta) + S_{k-2}(\alpha, \beta)). \end{aligned}$$

Thus, $Q_0(M_k)$ continues to an entire function of β for each k . Bearing in mind that the Satake parameter for π is the matrix $t_\chi = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \in GL(2, \mathbb{C})$, then $\text{Tr}(\vee^k t_\chi) = S_k(\alpha, \beta)$ for each $k \geq 0$, whence the theorem is established. $\square$

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