THE RICCATI FLOW AND SINGULARITIES OF SCHUBERT VARIETIES

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ABSTRACT. Let $\text{Gr}(m, n)$ be the Grassmannian of m -dimensional subspaces of an n -dimensional k -vector space, with $k = \mathbf{R}$ or $\mathbf{C}$. Fix an $n \times n$ matrix R with coefficients in k . The Riccati Flow Φ on $\text{Gr}(m, n)$ is the action of a one-parameter subgroup of $\text{GL}_n(k)$, given by $\Phi_t(\Lambda) = e^{Rt}\Lambda$. We prove:

Theorem. *Let X be a Schubert variety in $\text{Gr}(m, n)$. Then there exists a Riccati flow Φ on X and a stable manifold W for Φ such that W is the smooth locus of X .*

Corollary (over $\mathbf{C}$). *X as above is smooth if and only if the cohomology of X satisfies Poincaré Duality.*

1. INTRODUCTION

Schubert varieties and their singularities have been a focus of recent research in algebraic topology, algebraic geometry, representation theory, combinatorics, and systems and control theory (see [GM] for references). There are several ways to detect singularities: briefly, these are Kazhdan-Lusztig polynomials (representation theory), intersection cohomology (algebraic topology), Standard Monomial Theory (algebraic geometry), and “lexicographically” (combinatorics and algebraic geometry). This paper presents a new approach to the computation of the singular locus of a Schubert variety in a Grassmannian.

The basic tool in this approach is a certain one-parameter subgroup of the general linear group, called the *Riccati flow*; the reason for the name will be discussed below.

In the following, k denotes either the real or complex numbers; the *Grassmannian* $\text{Gr}(m, n)$ is the variety of m -dimensional subspaces of a vector space of dimension n over k ; and *cohomology* is singular cohomology with integral coefficients, unless explicitly stated otherwise.

The main result is

(1.1) **Theorem.** *Let X be a Schubert variety in $\text{Gr}(m, n)$. Then there exists a Riccati flow on $\text{Gr}(m, n)$ and a stable manifold W of this flow such that W is exactly the smooth locus of X .*

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This has the following interesting corollary, which should be compared with a result proved for general Schubert varieties by Deodhar [D] using Standard Monomial Theory. The present proof does not, however, use the representation theory of SL_n .

(1.2) **Corollary** (over $\mathbf{C}$). *Let X be a Schubert variety in $Gr(m, n)$. Then X is smooth if and only if the singular cohomology of X satisfies Poincaré Duality.*

The contents are a brief review of Schubert varieties (§2), Shayman's phase portrait for the Riccati flow (§3), and the proofs (§4).

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2. REVIEW OF SCHUBERT CELLS AND SCHUBERT VARIETIES

This section presents a brief review of Schubert cells and Schubert varieties. See [W] or [GeM] for details.

The Schubert cells form a cell decomposition of the Grassmannian $Gr(m, n)$ and are defined as follows. Let V_n be a vector space of dimension n over k . Fix a complete flag of subspaces

$$F_1 \subset F_2 \subset \cdots \subset F_n = V_n,$$

where $\dim(F_i) = i$. Choose m integers $d_1 < \cdots < d_m \leq n$. Define the *Schubert cell* $[d_1 d_2 \cdots d_m]$ by

$$[d_1 \cdots d_m] = \{\Lambda \in Gr(m, n) : \dim(\Lambda \cap F_{d_j}) = j, \dim(\Lambda \cap F_{d_j}) \neq j + 1\}.$$

The closure of $[d_1 \cdots d_m]$ in $Gr(m, n)$ is a *Schubert variety*; it is a complete irreducible projective variety of k -dimension $\Sigma(d_j - j)$. It is a fundamental fact that a Schubert variety is a disjoint union of Schubert cells.

A Schubert cell (or variety—context makes clear which is meant) can be represented by a *word* of length n constructed from the alphabet $\{a, b\}$. The letters in positions d_j , $j = 1, \dots, m$, are a , while the remaining letters are b . The string $aa \cdots a$ (k occurrences) is often abbreviated a^k . Thus, for example, the unique codimension 1 Schubert variety in $Gr(2, 4)$ can be represented as $[24]$ or by the word *baba*. It consists of those Λ which satisfy the two “Schubert conditions”

$$\dim(\Lambda \cap F_2) = 1, \quad \Lambda \neq F_2,$$

and

$$\dim(\Lambda \cap F_4) = 2, \quad \text{i.e., } \Lambda \subset F_4.$$

(Although the last condition is, in fact, vacuous.)

There is a partial order on the set of Schubert cells in $Gr(m, n)$, variously called the *Bruhat order*, the *adherence order*, or the *specialization order* (see [P] or [HM]). The easiest way to define it for Schubert cells C_1 and C_2 is by

$$C_1 \leq C_2 \Leftrightarrow C_1 \subset \overline{C_2},$$

where $\overline{X}$ denotes the closure of X . This order is generated by the “moves”

$$(2.1) \quad w_1 b a w_2 > w_1 a b w_2,$$

where w_1 and w_2 are arbitrary subwords. See [P].

The cells with this partial order define a lattice, called the *Hasse diagram*. For $\text{Gr}(2, 4)$ it is

where $w \rightarrow w'$ indicates that $w \subset \overline{w'}$.

3. THE RICCATI FLOW

Let R be an $(n+m) \times (n+m)$ matrix with entries in k , $k = \mathbf{C}$ or $k = \mathbf{R}$. The Riccati Flow Φ generated by R on $\text{Gr}(n, n+m)$ is given by

$$\Phi_t(\Lambda) = e^{Rt}\Lambda,$$

where $\Lambda \in \text{Gr}(n, n+m)$, e^{Rt} is the matrix exponential, $t \in k$, and the right-hand side is the usual action of $\text{GL}(n+m)$ on subspaces. The name comes from systems and control theory: there are local coordinates on the Grassmannian such that the flow is infinitesimally generated by the vector field

$$\frac{dK}{dt} = R_{21} + R_{22}K - KR_{11} - KR_{12}K$$

when the block decomposition for R is

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

This is the *matrix Riccati equation*, which is used in certain optimal control problems [B]. The flow can be extended to various flag manifolds in an obvious manner.

Control theorists are most interested in the case of a semisimple R whose eigenvalues have distinct real parts except those that occur as pairs of complex conjugates. Shayman [S] determined the phase portrait for the Riccati equation on the Grassmannian for semisimple R . (Although he worked over $\mathbf{R}$, the same proofs go through for $\mathbf{C}$ with minor modifications.) This involves finding what [S] refers to as the various *invariant loci* $T_1, \dots, T_r$ and *stable manifolds*

$$W_i = \{\Lambda: \lim_{t \rightarrow \infty} \Phi_t \Lambda \in T_i\}.$$

When R is semisimple, the flow is a $\mathbf{C}^*$ action on the Grassmannian.

(More generally, recent work of the author indicates that these results follow from the Schur-Kostant-Atiyah-Guillemin-Sternberg Convexity Theorem [A]. Also, compare this with the Bialynicki-Birula plus and minus decompositions for a scheme with torus action [BB]. Note that in the proof of (1.1) we will construct an R with the desired properties, so the extra assumptions are justified.)

Here are Shayman's results. Fix an $(n + m)$ -dimensional k -vector space V . Let $\mu_1 < \cdots < \mu_r$ be the real parts of the eigenvalues of R . Then V has the direct sum decomposition

$$V = E_1 \oplus \cdots \oplus E_r,$$

where each E_j is the direct sum of the primary components of R corresponding to those eigenvalues with real part μ_i ; let d_i be the dimension of E_i . Clearly E_i is R -invariant. Let $(l_1, \dots, l_r)$ be a partition of n with $l_j \leq d_j$. Define

$$M_j = E_1 \oplus \cdots \oplus E_j$$

and

$$T(l) = \text{Gr}(l_1, E_1) \times \cdots \times \text{Gr}(l_r, E_r).$$

(3.1) **Theorem** (Shayman). (i) For all l as above, $T(l)$ is invariant with respect to the Riccati flow;

(ii) its stable manifold is

$$W(l) = \left\{ \Lambda \in \text{Gr}(n, n + m) : \dim(\Lambda \cap M_s) = \sum_{j=1}^s l_j, \quad s = 1, \dots, r \right\}.$$

(3.2) **Proposition.** $W(l)$ is dense in the Schubert variety whose word is $w = w_1 w_2 \cdots w_r$, where $w_j = b^{d_j - l_j} a^{l_j}$, and $\overline{W(l)} = \overline{w}$.

Proof. We construct the word for the largest cell in $W(l)$; the proposition follows immediately.

The word decomposes into subwords w_j , each corresponding to the conditions imposed on an E_j . Let $s = \dim(M_j)$. If $\dim(E_j) = 1 = l_1$, the j th Schubert condition on Λ is $\dim(\Lambda \cap F_s) = 1$, so $w_j = a$. Similarly if $l_j = 0$, $w_j = b$. If $\dim(E_j) = 2$ and $l_j = 0$, the j th condition is $\dim(\Lambda \cap F_s) = 0$, so $w_1 = bb$; if $l_j = 1$, $w_1 = ba$; and if $l_j = 2$, $w_1 = aa$. Notice that if $l_j = 1$, there is an apparent ambiguity, because the subword ab (corresponding to the condition $F_1 \subset \Lambda$) also applies. However, the cell abw' is in the closure of baw' . Q.E.D.

(3.3) **Example.** Let $r = 3$, and choose E_j such that $\dim(E_j) = 2$ ($j = 1, 2, 3$), so $\dim(M_1) = 2$, $\dim(M_2) = 4$, and $\dim(M_3) = 6$. Suppose $l = (1, 0, 2)$. Then the Λ in $W(l)$ satisfy the conditions $\dim(\Lambda \cap M_1) = 1$, $\dim(\Lambda \cap M_2) = 1$, and $\dim(\Lambda \cap M_3) = 3$, and the word is $babbba$. Similarly if $l = (1, 1, 1)$, the word is $bababa$.

4. PROOFS

To begin, choose a Schubert variety X in $\text{Gr}(n, n + m)$. The proof of (1.1) involves constructing a partition $(d_1, \dots, d_n)$ of $n + m$, constructing a matrix R with primary components E_j of dimension d_j , and choosing a sequence $(l_1, \dots, l_n)$ of nonnegative integers with $l_j \leq d_j$.

Partition $n + m$ into n integers as follows: let $w_1, \dots, w_n$ be the subwords of the word for X which end with a . For example, if $X = bbaabaa$, then $n = 4$ and $w_1 = bba$, $w_2 = a$, $w_3 = ba$, and $w_4 = a$. Let d_j be the number of letters in w_j . The desired partition is $(d_1, \dots, d_n)$.

Define the matrix R with n Jordan blocks of size d_j having eigenvalue $-j$ (the actual value of the eigenvalue is not important; the choice here is just for reference). Let E_j be the subspace of k^{n+m} corresponding to the j th block. Finally, let $l_j = 1$, i.e., let $W = W(1, \dots, 1)$. It is clear that the largest cell in W is the largest cell in X . W is the union of cells whose words consist of n subwords v_j of length d_j containing a single a ,

Informally, cells in W are named by moving an a within a subword but never by moving it into a different subword. When an a moves into a different subword, the word so created represents a cell in the closure of W but not in W . These cells are "just below" W in the Hasse diagram. Call these cells "just below" W the *shear zone*. For example, consider W given by $bababa$. The shear zone consists of $aabbba$ and $baaabb$. The cell $aabbab$ is of codimension 1 in the closure of $ababab$, a cell in the closure of $bababa$, and so might appear to be in the shear zone; however, it is already in the closure of a shear zone cell, e.g., $aabbba$.

This concept simplifies the statement of the main theorem:

(1.1, bis) **Theorem.** *The singular locus of X is the closure of the shear zone.*

(4.1) **Lemma** (Characterization of the Shear Zone). *The shear zone is the set of cells made up of n subwords v_j of length d_j , exactly one of which contains two occurrences of the letter a . In other words, cells in the shear zone have the form*

$$w_1 w_2 \cdots w'_s w'_{s+1} \cdots w_n,$$

where $w'_s = b^{d_s-2}aa$ and $w'_{s+1} = b^{d_{s+1}}$.

Thus, if C is a cell in the shear zone, its word differs from that of W in at most two subwords: the one with two as , and the one with no as .

We now proceed by induction on the dimension of X . It is trivial to check that all Schubert varieties of dimension 1 or 2 are nonsingular and, in these cases, that $W = X$. Similarly, there are four families of Schubert varieties of dimension 3: $abbaB$, $baaaB$, $bbbaB$, and $babaB$ (here, B indicates a string of arbitrary length of the letter b). Members of the first three are nonsingular, and all satisfy $W = X$; those in the last are singular. The Hasse diagram of such an X is

while W consists of all these cells except $aabbB$. It can be verified directly that $aabbB$ is the singular locus. Thus, the theorem is true in the case of dimension 3 or less.

Now, restrict attention to the shear zone in the cell decomposition of X . If $n = 2$, then there are two subwords. The shear zone consists of the cell $b^{d_1-2}aab^{d_2}$. It is easy to verify directly that the closure of this cell is the singular locus (for example, write down a parametrization of X).

If $n > 3$, then the word for a cell in the shear zone differs from that of W in two subwords: the one with aa , and the one with no as . There must therefore exist, for each cell C in the shear zone, an integer r ($0 < r < n$) such that one of the following conditions holds:

- (L) the first r subwords of C are identical to the first r subwords of W ;
or
(R) the last r subwords of C are identical to the last r subwords of W .

Case (R). Consider the fibering $\phi: \text{Gr}(n, n+m) \rightarrow \text{Gr}(n-1, n+m-d_1)$ given by

$$\phi(\Lambda) := (\Lambda/(\Lambda \cap E_1) \subset (V/E_1));$$

in local coordinates, ϕ is a projection between vector spaces. Then $\phi(C)$ is in the shear zone of some stable manifold W' for the flow on $\text{Gr}(n-1, n+m-d_1)$ corresponding to the lower right $(n+m-d_1) \times (n+m-d_1)$ submatrix of R ; this follows immediately from the combinatorial characterization of the shear zone. Since the image cell is in the singular locus of the closure of W' and since ϕ is locally a projection, C is in the singular locus of the closure of W .

Case (L). In this case, define

$$E = E_1 \oplus \cdots \oplus E_{n-1},$$

and let $d = \dim(E)$. Define a map $\text{Gr}(n, n+m) \rightarrow \text{Gr}(n-1, d)$ by

$$\Lambda \mapsto (\Lambda \cap E) \subset (V \cap E).$$

It again follows from (4.1) that the image of a cell in the shear zone is a cell in the shear zone of a stable manifold of a flow on the image Grassmannian. The rest of the argument is similar.

Proof of the Corollary. W constructed above is nonsingular, and hence its cohomology satisfies Poincaré duality: if $N = \dim(W)$, then $\text{rank}(H^{N-i}(W)) = \text{rank}(H^i(W))$. If the cohomology of X does not satisfy Poincaré duality, then there are cells in X not in W . These all occur in or below the "shear zone". The complement of W is the singular locus and is nonempty. Q.E.D.

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