STOCHASTIC CONTINUITY OF RANDOM DERIVATIONS ON H^* -ALGEBRAS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper we obtain the continuity of stochastically derivative linear random operators on (nonassociative) H^* -algebras with zero annihilator. Moreover, we investigate the stochastic size of the separating subspace for linear random operators, on H^* -algebras, which have some probability of being derivative.

I. INTRODUCTION

The continuity of derivations is a basic automatic continuity problem which has attracted many authors' attention since the Johnson-Sinclair theorem [18], in 1968, asserted that *derivations on semisimple Banach algebras are continuous*; the most crucial result in this direction. Even in the nonassociative context, a great interest has been taken in these problems [17, 30, 31, 34, 35].

Random operator on Banach algebras are of lively interest for their intimate connections with the random equations theory [4, 25], and stochastically derivative random operators seem to be a new topic which was considered in [29]. Starting with the Johnson-Sinclair theorem, the notion of stochastically derivative operator was introduced, and a random version of the quoted theorem, showing that *stochastically derivative linear random operators on semisimple Banach algebras are stochastically continuous*, was proved.

Now we shall be concerned with random derivations on (nonassociative) H^* -algebras.

In this paper we show a random generalization of the fundamental result in [31] which asserts the continuity of derivations on H^* -algebras with zero annihilator. For that, we say that a linear random operator, D , on a possibly nonassociative normed algebra, A , is *stochastically derivative* if

$$\mathbb{P}[D(ab) = D(a)b + aD(b)] = 1 \quad \forall a, b \in A,$$

and we show that *stochastically derivative linear random operators on H^* -algebras with zero annihilator are (stochastically) continuous*. In fact, we can measure discontinuity of D by considering its so-called *separating subspace*, and we obtain, in Theorem 7, that *every element in the separating subspace for*

Received by the editors April 5, 1993 and, in revised form, June 9, 1993.
 1991 *Mathematics Subject Classification*. Primary 46H40, 47B80.

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 0002-9939/95 \$1.00 + \$.25 per page

a stochastically derivative linear random operator on an H^* -algebra A is almost surely valued in the annihilator of A .

Moreover, we examine the stochastic continuity property for a linear random operator, D , on an H^* -algebra A , which has a slight probability of being stochastically derivative in the following sense:

$$\delta(D) = \inf\{\mathbb{P}[D(ab) = D(a)b + aD(b)] : a, b \in A\} > 0$$

and, in Corollary 2, we investigate the “stochastic size” of the separating subspace for D , showing that every element in the separating subspace for D is valued in the annihilator of A with probability at least δ .

II. SOME RESULTS ABOUT H^* -ALGEBRAS

H^* -algebras introduced and studied by Ambrose [1] in the associative case have been considered also in the case of the most familiar classes of nonassociative algebras [6, 8, 10, 11, 21–23, 32, 33] and even in the general nonassociative context [9, 11, 12].

As it is known, an H^* -algebra is a complex algebra A with a (conjugate-linear) algebra involution $*$ whose underlying vector space is a Hilbert space in which the equalities

$$(ab|c) = (a|cb^*) = (b|a^*c)$$

hold for all a, b , and c in A .

The annihilator, $\text{Ann}(A)$, of an algebra A is defined as the set of those a in A such that $ab = ba = 0$ for every b in A , and a familiar result in the theory of H^* -algebras is the following.

Theorem 1. *Every H^* -algebra is the orthogonal sum of its annihilator and an H^* -algebra with zero annihilator.*

As it is well known, a normed algebra is said to be *topologically simple* if it has nonzero product but not nonzero proper closed ideals. The next theorem is a crucial result in the theory of nonassociative H^* -algebras.

Theorem 2 [11]. *Every H^* -algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, and these are topologically simple H^* -algebras.*

As in [31] we apply techniques of central closeability in our study on continuity of “random derivations” on H^* -algebras.

We recall that a prime algebra A over a field $\mathbb{K}$ is said to be *centrally closed* if, for every nonzero ideal I of A and for every linear mapping $f : I \rightarrow A$ satisfying $f(ax) = af(x)$ and $f(xa) = f(x)a$ for all a in A and x in I , there exists λ in $\mathbb{K}$ such that $f(x) = \lambda x$ for all x in I (cf. [13]).

Examples of centrally closed algebras are the topologically simple H^* -algebras, as the next theorem shows.

Theorem 3 [9]. *Every topologically simple H^* -algebra is a centrally closed prime algebra.*

For an element a in the algebra A , we denote by $\mathcal{L}_a$ and $\mathcal{R}_a$ the operators of left and right multiplication by a on A , respectively. The usual multiplication algebra of A will be denoted by $M(A)$. Since for an element a in an

H^* -algebra A the adjoint L_a^* of the operator L_a is L_{a^*} and the adjoint of R_a is R_{a^*} , for every T in $M(A)$ we have that T^* lies in $M(A)$.

Following the pattern established in [31], our next goal consists of providing suitable sequences for the “random continuity problem”. For that, we will need Theorem 3.1 of [13] which we establish now.

Theorem 4 [13]. *Given linearly independent elements $x_1, \dots, x_n$ in a centrally closed prime algebra A , there exists T in $M(A)$ such that $T(x_1) = \dots = T(x_{n-1}) = 0$ and $T(x_n) \neq 0$.*

Lemma 1. *Let A be a topologically simple algebra and G a nonempty open subset of A which does not contain the zero. If T is in $M(A)$ and $\dim T(A) = 1$, then there exists R in $M(A)$ such that*

$$(RT)^2(A) \cap G \neq \emptyset.$$

Proof. Let b be in A such that $T(A) = \text{lin}\{b\}$, and let a be in A such that $T(a) = b$. Since $T \neq 0$, we have that the open subset G is not contained in $\text{Ker } T$, so $[A \setminus \text{Ker } T] \cap G$ is a nonempty open subset of A . Now the topological simplicity of A assures the existence of R in $M(A)$ such that $R(b) \in [A \setminus \text{Ker } T] \cap G$. Since $R(b) \notin \text{Ker } T$, we have that $R(b) = \alpha a + x$, where $\alpha \in \mathbb{K} \setminus \{0\}$ and x lies in $\text{Ker } T$, so that

$$(RT)^2(\alpha^{-1}a) = \alpha^{-1}(RTR)b = \alpha^{-1}(RT)(\alpha a + x) = \alpha^{-1}R(\alpha b) = R(b) \in G. \quad \square$$

Finally we use central closeability and topological simplicity to obtain sequences of great importance in this work.

Theorem 5. *Let A be a centrally closed topologically simple algebra with infinite dimension and G a nonvoid open subset of A which does not contain the zero. Then one of the following assertions holds:*

- (1) *There exist sequences $\{a_n\}$ in A and $\{T_n\}$ in $M(A)$ such that:*

$$T_n \cdots T_1 a_n \in G \text{ and } T_{n+1} T_n \cdots T_1 a_n = 0 \quad \forall n \in \mathbb{N}.$$

- (2) *There exists a sequence $\{S_n\}$ in $M(A)$ such that:*

$$\dim S_n(A) = 1,$$

$$S_n^2(A) \cap G \neq \emptyset,$$

$$S_n S_m = 0 \quad \text{if } m < n.$$

Proof. We assume that assertion (2) does not hold, and we deduce (1) from this assumption.

In a first step we prove that (1) is true if $\dim T(A) \geq 2$ for all T in $M(A) \setminus \{0\}$. By the topological simplicity of A , we obtain a_1 in A and T_1 in $M(A)$ such that $T_1 a_1 \in G$ and suppose that $a_1, \dots, a_k$ and $T_1, \dots, T_k$ have been chosen inductively such that $T_j \cdots T_1 a_{j-1} = 0$ and $T_j \cdots T_1 a_j \in G$ for $j = 2, \dots, k$. Since $\dim(T_k \cdots T_1)(A) \geq 2$, there exists a_{k+1} in A such that $T_k \cdots T_1 a_k$ and $T_k \cdots T_1 a_{k+1}$ are linearly independent, so (by Theorem 4) there exists T in $M(A)$ such that $TT_k \cdots T_1 a_k = 0$ and $TT_k \cdots T_1 a_{k+1} \neq 0$. Again by the topological simplicity of A we obtain now S in $M(A)$ such that $STT_k \cdots T_1 a_{k+1} \in G$ and we define $T_{k+1} = ST$. The sequences $\{a_n\}$ and $\{T_n\}$ constructed in this way satisfy the requirements in the first assertion.

Now we assume that there exists an element T in $M(A)$ such that $\dim T(A) = 1$, and we apply the above lemma to obtain an element S in $M(A)$ such that $\dim S(A) = 1$ and $S^2(A) \cap G \neq \emptyset$. Let $\mathcal{V}$ be the set of all finite or infinite sequences $\{S_n\}_{n=1}^N$ of elements in $M(A)$ satisfying:

$$\begin{aligned} \dim S_n(A) &= 1, & n &= 1, \dots, N; \\ S_n^2(A) \cap G &\neq \emptyset, & n &= 1, \dots, N; \\ S_n S_m &= 0, & 1 \leq m < n \leq N. \end{aligned}$$

Since the sequence $\{S\}$ lies in $\mathcal{V}$, we have that this set is nonempty. Moreover, we consider in $\mathcal{V}$, the partial order defined by

$$\{S_n\}_{n=1}^N \leq \{S'_m\}_{m=1}^M \quad \text{iff } N \leq M \text{ and } S_n = S'_n \text{ for } n = 1, \dots, N.$$

Then $\mathcal{V}$ is an inductive set, so the Zorn lemma assures the existence of a maximal element $\{S_n\}_{n=1}^N$ in $\mathcal{V}$ which has $N < \infty$ (because we assume that (2) does not hold). For every $n = 1, \dots, N$ let b_n be in A such that $S_n(A) = \text{lin}\{b_n\}$ (first condition) and $S_n(b_n) \neq 0$ (second condition). From the equality $S_n S_m = 0$, $1 \leq m < n \leq N$, we deduce now that vectors $\{b_1, \dots, b_N\}$ are linearly independent. Now, let a_1 be in A such that $\{b_1, \dots, b_N, a_1\}$ are linearly independent, and let T_1 be in $M(A)$ such that

$$T_1(b_1) = \dots = T_1(b_N) = 0 \quad \text{and} \quad T_1(a_1) \neq 0.$$

If $\dim T_1(A) = 1$, we apply the previous lemma to obtain again an element R in $M(A)$ such that the element $S_{N+1} = RT_1$ has the property

$$S_{N+1}^2(A) \cap G \neq \emptyset,$$

but trivially

$$\dim(S_{N+1})(A) = 1,$$

and

$$S_{N+1} S_n = 0, \quad n = 1, \dots, N.$$

Then the sequence $\{S_n\}_{n=1}^{N+1}$ lies in $\mathcal{V}$ which contradicts that $\{S_n\}_{n=1}^N$ is maximal. Therefore, it must be $\dim T_1(A) \geq 2$, and thus there exists a_2 in A such that $T_1(a_1)$ and $T_1(a_2)$ are linearly independent. Now let T_2 be in $M(A)$ such that

$$T_2 T_1(a_1) = 0 \quad \text{and} \quad T_2 T_1(a_2) \in G.$$

Reasoning as above we deduce that $\dim(T_2 T_1)(A) \geq 2$. We suppose that $a_1, \dots, a_k$ and $T_1, \dots, T_k$ has been chosen inductively such that

$$T_j \cdots T_1 a_{j-1} = 0,$$

$$T_j \cdots T_1 a_j \in G \quad \text{and} \quad \dim(T_j \cdots T_1)(A) \geq 2 \text{ for } j = 2, \dots, k,$$

and we get a_{k+1} in A such that $(T_k \cdots T_1)(a_k)$ and $(T_k \cdots T_1)(a_{k+1})$ are linearly independent. Therefore, by Theorem 4 and the topological simplicity of A , there exists T_{k+1} in $M(A)$ such that $T_{k+1} \cdots T_1 a_k = 0$ and $T_{k+1} \cdots T_1 a_{k+1} \in G$. If $\dim(T_{k+1} \cdots T_1)(A) = 1$, we apply Lemma 1 to obtain again an element $S_{N+1} = RT_{k+1} \cdots T_1$ in $M(A)$ so that $S_{N+1}^2(A) \cap G \neq \emptyset$ and clearly $\dim(S_{N+1})(A) = 1$ and $S_{N+1} S_n = 0$, $n = 1, \dots, N$. So it must be $\dim(T_{k+1} \cdots T_1)(A) \geq 2$. Sequences $\{a_n\}$ and $\{T_n\}$ constructed in this way satisfy assertion (1). $\square$

Since by Theorem 3 topologically simple H^* -algebras are centrally closed, we obtain for such an algebra in the following result.

Corollary 1. *Let A be an infinite-dimensional topologically simple H^* -algebra and G a nonempty open subset of A not containing the zero. Then, one of the following assertions holds:*

- (1) *There exist sequences $\{a_n\}$ in A and $\{T_n\}$ in $M(A)$ such that:*

$$T_n \cdots T_1 a_n \in G \text{ and } T_{n+1} T_n \cdots T_1 a_n = 0 \quad \forall n \in \mathbb{N}.$$

- (2) *There exists a sequence $\{S_n\}$ in $M(A)$ such that:*

$$\dim S_n(A) = 1,$$

$$S_n^2(A) \cap G \neq \emptyset,$$

$$S_n S_m = 0 \quad \text{if } m < n.$$

III. STOCHASTICALLY DERIVATIVE LINEAR RANDOM OPERATORS

The study of Banach algebra-valued random variables is of great importance in the random equations theory [4, 25]. For this reason, these have been considered extensively, mainly for C^* -algebras and associative H^* -algebras [4, 14, 15, 24, 25].

For a Banach space X and a probability space $(\Omega, \Sigma, \mathbb{P})$ we consider the linear space $\mathcal{L}_0(\Omega, X)$ ($\mathcal{L}_0(X)$ if there is no confusion) of all X -valued Bochner random variables on $(\Omega, \Sigma, \mathbb{P})$, endowed with the usual probability convergence topology (cf. [20]). As it is pointed out in [20] with the almost surely identification on $\mathcal{L}_0(X)$ we obtain a complete metrizable topological linear space denoted by $L_0(\Omega, X)$ ($L_0(X)$ if there is no confusion).

Since the operation of multiplication on an H^* -algebra A is continuous, we have that the linear space $\mathcal{L}_0(A)$ is an algebra which contains A as a subalgebra.

We recall that a random operator T from a Banach space X to a Banach space Y is an operator from X to $\mathcal{L}_0(Y)$, and it is called linear if, for all x and y in X and all α and β in $\mathbb{K}$,

$$\mathbb{P}[T(\alpha x + \beta y) = \alpha T x + \beta T y] = 1,$$

i.e., T is linear when we consider it as an operator from X into $L_0(Y)$.

Moreover, the continuity of T , also called stochastic continuity, means that, for every x in X ,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\|Tx_n - Tx\| \geq \varepsilon] = 0,$$

for every sequence $\{x_n\}$ in X converging to x and every positive ε , i.e., the usual continuity when T is regarded as an $L_0(Y)$ -valued operator.

We show in [27] that stochastic continuity of a linear random operator T between Banach spaces X and Y depends directly on the stochastic size of its separating subspace $\mathcal{S}(T)$ where, as it is usual,

$$\mathcal{S}(T) = \{y \in \mathcal{L}_0(Y) : \exists \{x_n\} \rightarrow 0 \text{ in } X \text{ with } \{Tx_n\} \rightarrow y \text{ in probability}\}.$$

For a linear random operator T and a measurable subset Ω' of Ω , with $\mathbb{P}[\Omega'] > 0$, we consider the conditional operator $T_{\Omega'} : X \rightarrow \mathcal{L}_0(\Omega', Y)$ given by $T_{\Omega'}(x) = T(x)|_{\Omega'}$, and following [27] we say that T is probably continuous if some of its conditional operators are stochastically continuous. For such an operator we define the number

$$c(T) = \sup\{\mathbb{P}[\Omega'] : T_{\Omega'} \text{ is stochastically continuous}\},$$

and also we define $c(T) = 0$ to be zero if T is not probably continuous. This number may be considered as “the probability of being T stochastically continuous”, and it was shown, in [27, Corollary 8], as the following result.

Lemma 2. *If T is a linear random operator between Banach spaces, then*

$$c(T) = \lim_{\varepsilon \rightarrow 0} \lim_{x \rightarrow 0} \mathbb{P}[\|T(x)\| \leq \varepsilon].$$

For an H^* -algebra A and an element a in $\mathcal{L}_0(A)$, we denote by $\mathcal{L}_a$ and $\mathcal{R}_a$ the random operators of left and right multiplication by a on A , respectively, which are obviously stochastically continuous linear random operators on A .

Note that if R and S are bounded linear operators on Banach spaces X and Y , respectively, and T is a linear random operator from X to Y , then the operators TS and RT defined on X by

$$(TS)(x) = T(Sx)$$

and

$$(RT)(x) = R \circ (Tx)$$

(respectively) are linear random operators from X into Y that are continuous if T is. Thus, the set of all continuous random operators from X into Y is a right $BL(X)$ -module and a left $BL(Y)$ -module.

Now, for an H^* -algebra A , we define the random multiplication module on A , denoted by $\mathcal{RM}(A)$, as the two sided $M(A)$ -submodule of linear random operators on A generated by the identity and the random multiplication operators on A . It is clear that every random operator in $\mathcal{RM}(A)$ is stochastically continuous.

A linear random operator D on a (possibly nonassociative) normed algebra A is called *stochastically derivative* if, for all elements a and b in A ,

$$\mathbb{P}[D(ab) = D(a)b + aD(b)] = 1.$$

If D is a stochastically derivative linear random operator on a normed algebra A , then we have $DL_a - L_aD = \mathcal{L}_{D(a)}$ and $DR_a - R_aD = \mathcal{R}_{D(a)}$; hence, for every T in $M(A)$ the random operator $d(T) = DT - TD$ lies in $\mathcal{RM}(A)$ and so it is stochastically continuous.

Now we use the sequences obtained in Corollary 8 to obtain a stochastic continuity property of stochastically derivative linear random operators on H^* -algebras with zero annihilator.

Proposition 1. *Let D be a stochastically derivative linear random operator on an H^* -algebra with zero annihilator A , and let I be a minimal closed ideal of A . Then, for every b in I , the linear random functional D_b on A defined by $a \mapsto (D(a)|b)$ is stochastically continuous.*

Proof. Step 1. In the first step we assume that $\dim I < \infty$.

Thanks to the stochastic Banach-Steinhaus theorem [28], it is straightforward to show that the set

$$\mathcal{E}(I) = \{b \in I : D_b \text{ is stochastically continuous}\}$$

is a closed ideal of I . Therefore, from the topological simplicity of I (Theorem 3), it follows that $\mathcal{E}(I) = 0$ or $\mathcal{E}(I) = I$.

Since

$$\begin{aligned}(D(a)|b_1b_2) &= (b_1^*D(a)|b_2) = (D(b_1^*a)|b_2) + (D(b_1^*)a|b_2) \\ &= (D_{b_2}L_{b_1^*})(a) + (\mathcal{L}_{D(b_1^*)}a|b_2)\end{aligned}$$

almost surely for all b_1, b_2 in I and a in A , we have that $D_{b_1b_2}$ is stochastically continuous and so $b_1b_2 \in \mathcal{E}(I)$. This proves that $(I^2)^-$ is contained in $\mathcal{E}(I)$ and thus $\mathcal{E}(I) = I$ (because $(I^2)^- = I$).

Step 2. Finally we assume that $\dim I = \infty$.

If, on the contrary, there exists b_0 in I such that D_{b_0} is not stochastically continuous, then there exists a positive ε with $c(D_{b_0}) < \varepsilon$, and we consider the nonempty set

$$G = \{b \in I : c(D_b) < \varepsilon\},$$

which, by the random Banach-Steinhaus theorem [28], is open. Thus by Corollary 1 we obtain either of the following assertions:

(i) There exist sequences $\{a_n\}$ in I and $\{T_n\}$ in $M(I)$ such that:

$$T_n \cdots T_1 a_n \in G \text{ and } T_{n+1} T_n \cdots T_1 a_n = 0 \quad \forall n \in \mathbb{N}.$$

(ii) There exists a sequence $\{S_n\}$ in $M(I)$ such that:

$$\dim S_n(A) = 1,$$

$$S_n^2(A) \cap G \neq \emptyset,$$

$$S_n S_m = 0 \quad \text{if } m < n.$$

In the first place we assume that (i) is satisfied. Since $d(T_1^* \cdots T_n^*)$ is stochastically continuous, there exists a sequence $\{\mu_n\}$ of positive numbers such that, for every b in A with $\|b\| \leq 1$,

$$\mathbb{P}[\|d(T_1^* \cdots T_n^*)(b)\| \leq \mu_n] \geq 1 - \frac{1}{3n}$$

for all n in $\mathbb{N}$.

Now, using that $c(D_{T_n \cdots T_1 a_n}) < \varepsilon$, by Lemma 2 we can build inductively a sequence $\{b_n\}$ in A and a sequence $\{\nu_n\}$ of positive numbers with the property that for all n in $\mathbb{N}$ we have

$$\|b_n\| \leq 2^{-n},$$

$$\mathbb{P}\left[\left|\sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* b_j)|a_n)\right| \leq \nu_n\right] \geq 1 - \frac{1}{3n},$$

$$\mathbb{P}[(D(b_n)|T_n \cdots T_1 a_n)| \geq n + \nu_n + \mu_n + \mu_{n+1}] \geq 1 - \varepsilon.$$

Now we consider the element b in A defined by

$$b = \sum_{j=1}^{\infty} T_1^* \cdots T_j^* b_j,$$

and for n in $\mathbb{N}$ we write $c_n = b_{n+1} + \sum_{j=n+2}^{\infty} T_{n+2}^* \cdots T_j^* b_j$. As in the proof of Theorem 4, in [31], we have

$$\begin{aligned}(D(b)|a_n) &= (D(b_n)|T_n \cdots T_1 a_n) + \sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* b_j)|a_n) \\ &\quad + (d(T_1^* \cdots T_n^*)(b_n)|a_n) + (d(T_1^* \cdots T_{n+1}^*)(c_n)|a_n).\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}[||D(b)|| \geq n] &\geq \mathbb{P}[|(D(b)|a_n)| \geq n] \\
&\geq \mathbb{P}\left[|(D(b_n)|T_n \cdots T_1 a_n)| - \left|\sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* b_j)|a_n)\right| \right. \\
&\quad \left. - |(d(T_1^* \cdots T_n^*)(b_n)|a_n)| - |(d(T_1^* \cdots T_{n+1}^*)(c_n)|a_n)| \geq n\right] \\
&\geq \mathbb{P}\left[|(D(b_n)|T_n \cdots T_1 a_n)| \geq n + \left|\sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* b_j)|a_n)\right| \right. \\
&\quad \left. + ||d(T_1^* \cdots T_n^*)(b_n)|| + ||d(T_1^* \cdots T_{n+1}^*)(c_n)||\right] \\
&\geq \mathbb{P}[|(D(b_n)|T_n \cdots T_1 a_n)| \geq n + \nu_n + \mu_n + \mu_{n+1}] \\
&\quad + \mathbb{P}\left[\left|\sum_{j=1}^{n-1} (D(T_1^* \cdots T_j^* b_j)|a_n)\right| \leq \nu_n\right] - 1 \\
&\quad + \mathbb{P}[||d(T_1^* \cdots T_n^*)(b_n)|| \leq \mu_n] - 1 \\
&\quad + \mathbb{P}[||d(T_1^* \cdots T_{n+1}^*)(c_n)|| \leq \mu_{n+1}] - 1.
\end{aligned}$$

So that, for all n in $\mathbb{N}$,

$$\mathbb{P}[||D(b)|| \geq n] \geq 1 - \varepsilon - \frac{1}{3n} - \frac{1}{3n} - \frac{1}{3(n+1)} \geq 1 - \frac{1}{n} - \varepsilon,$$

and thus $\mathbb{P}[||D(b)|| = \infty] \geq 1 - \varepsilon$. This contradiction shows the stochastic continuity for every random functional D_b when (i) is satisfied.

Finally we assume that (ii) holds, i.e., there exists a sequence $\{S_n\}$ in $M(I)$ such that $\dim S_n(I) = 1$, $S_n^2(I) \cap G \neq \emptyset$, and $S_n S_m = 0$ if $m < n$. Since the random operators $d((S_n^2)^*)$ and $d(S_n^*)$ are stochastically continuous, there exists a sequence $\{\mu_n\}$ of positive numbers such that, for every b in A with $||b|| \leq 1$,

$$\mathbb{P}[||d((S_n^2)^*)(b)|| \leq \mu_n] \geq 1 - \frac{1}{3n}$$

and

$$\mathbb{P}[||d(S_n^*)(b)|| \leq \mu_n] \geq 1 - \frac{1}{3n}$$

for all n in $\mathbb{N}$. Now for every natural number n there exists an element a_n in I such that $c(D_{S_n^2}(n)) < \varepsilon$, and so, using again Lemma 2, we can construct inductively a sequence $\{b_n\}$ in A and a sequence $\{\nu_n\}$ of positive numbers satisfying that, for every n in $\mathbb{N}$,

$$\begin{aligned}
||b_n|| &\leq ||S_n||^{-1} 2^{-n}, \\
\mathbb{P}\left[\left|\sum_{j=1}^{n-1} (D(S_n^* S_j^* b_j)|a_n)\right| \leq \nu_n\right] &\geq 1 - \frac{1}{3n}, \\
\mathbb{P}[|(D(b_n)|S_n^2 a_n)| \geq n ||S_n a_n|| + \nu_n + 2\mu_n] &\geq 1 - \varepsilon.
\end{aligned}$$

We consider b in A defined by $b = \sum_{j=1}^{\infty} S_j^* b_j$. It is easy to check that $S_n^* b = \sum_{j=1}^n S_n^* S_j^* b_j$, and so we have the following almost surely equalities:

$$\begin{aligned} (D(b)|S_n a_n) &= (S_n^* D(b)|a_n)(D(S_n^* b) - d(S_n^*)b|a_n) \\ &= \sum_{j=1}^{n-1} (D(S_n^* S_j^* b_j)|a_n) + (D((S_n^2)^* b)|a_n) - (d(S_n^*)b|a_n) \\ &= \sum_{j=1}^{n-1} (D(S_n^* S_j^* b_j)|a_n) + (d((S_n^2)^* b_n)|a_n) \\ &\quad + (D(b_n)|S_n^2 a_n) - (d(S_n^*)b|a_n). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}[||D(b)|| \geq n] &\geq \mathbb{P}[|(D(b)|S_n a_n)| \geq n||S_n a_n||] \\ &\geq \mathbb{P}\left[|(D(b_n)|S_n^2 a_n)| \geq n||S_n a_n|| + \left|\sum_{j=1}^{n-1} (D(S_n^* S_j^* b_j)|a_n)\right| \right. \\ &\quad \left. + |(d((S_n^2)^* b_n)|a_n)| + |(d(S_n^*)b|a_n)|\right] \\ &\geq \mathbb{P}[|(D(b_n)|S_n^2 a_n)| \geq n||S_n a_n|| + \nu_n + 2\mu_n] \\ &\quad + \mathbb{P}\left[\left|\sum_{j=1}^{n-1} (D(S_n^* S_j^* b_j)|a_n)\right| \leq \nu_n\right] - 1 \\ &\quad + \mathbb{P}[|d((S_n^2)^* b_n)| \leq \mu_n] - 1 + \mathbb{P}[|d(S_n^*)b| \leq \mu_n] - 1 \\ &\geq 1 - \varepsilon - \frac{1}{3n} - \frac{1}{3n} - \frac{1}{3n} \\ &= 1 - \frac{1}{n} - \varepsilon. \end{aligned}$$

This implies that $\mathbb{P}[||D(b)|| = \infty] \geq 1 - \varepsilon$, and this contradiction proves the stochastic continuity for every random functional D_b . $\square$

Now we obtain the continuity of D from the continuity of every D_b , using [29, Lemma 3.1].

Theorem 6. *Let D be a stochastically derivative linear random operator on an H^* -algebra A with zero annihilator. Then D is stochastically continuous.*

Proof. Let $\mathbf{a}$ be in the separating subspace $\mathcal{S}(D)$ for D . By Proposition 1, we have that, for every minimal closed ideal I of A and every element b in I , $\mathbb{P}[(a|b) = 0] = 1$. If we denote the set of all minimal closed ideal of A by $\mathcal{J}$ and for every I in $\mathcal{J}$ we denote the orthogonal complement of I in A by $I^\perp$, it follows from [29, Lemma 3.1] that

$$\mathbb{P}[a = 0] = \mathbb{P}[a \in \bigcap_{I \in \mathcal{J}} I^\perp] = \inf \left\{ \mathbb{P}[a \in \bigcap_{I \in \mathcal{F}} I^\perp] : \mathcal{F} \subset \mathcal{J} \text{ finite} \right\} = 1,$$

so [27, Theorem 3] proves that D is stochastically continuous. $\square$

Theorem 7. *Let D be a stochastically derivative linear random operator on an H^* -algebra A . Then every element in the separating subspace for D is almost surely valued in the annihilator of A .*

Proof. By Theorem 1, A is the orthogonal sum of the annihilator $\text{Ann}(A)$ and an H^* -algebra B with zero annihilator. Now, if we denote the orthogonal projection from A onto B by π_B , we have that the linear random operator $\pi_B D|_B$ on B is stochastically derivative so that, by the above theorem, it is stochastically continuous. $D(a)$ is also a random variable on A , almost surely valued on $\text{Ann}(A)$, for every a in $\text{Ann}(A)$.

Now let a be an element in the separating subspace for D , and let $\{a_n\}$ be a sequence in A such that $\lim a_n = 0$ and $\{D(a_n)\}$ converges in probability to a . Then, since

$$\pi_B D(a_n) = \pi_B D(\pi_B a_n) = (\pi_B D|_B)(\pi_B a_n)$$

for all n in $\mathbb{N}$, we have that $\mathbb{P}[\pi_B a = 0] = 1$, so $\mathbb{P}[a \in \text{Ann}(A)] = 1$. $\square$

IV. PROBABLY DERIVATIVE LINEAR RANDOM OPERATORS

Throughout this section we consider a linear random operator D on an H^* -algebra A which has only some probability of being derivative, in the following sense:

$$\inf\{\mathbb{P}[D(ab) = D(a)b + aD(b)] : a, b \in A\} = \delta(D) > 0.$$

Operators satisfying the above property will be called *probably derivative*.

A straightforward application of the multilinear uniform randomization principle [29, Theorem 4.1] to the bilinear random operator on A ,

$$(a, b) \mapsto D(ab) - D(a)b - aD(b),$$

shows the existence of a measurable subset Ω' with $\mathbb{P}[\Omega'] = \delta(D)$ such that, for every a and b in A ,

$$D(ab) = D(a)b + aD(b) \quad \text{almost surely on } \Omega',$$

i.e., the conditional operator $D_{\Omega'}$ is stochastically derivative. Now we can apply Theorem 7 to obtain the following result.

Corollary 2. *Let D be a probably derivative linear random operator on an H^* -algebra A , with a probability δ of being derivative. Then every element in the separating subspace for D is valued in the annihilator of A with a probability at least δ .*

Therefore, if the H^* -algebra A has zero annihilator, then D has a stochastically continuous conditional operator $D_{\Omega'}$ with $\mathbb{P}[\Omega'] = \delta(D)$, so that the next corollary follows.

Corollary 3. *Every probably derivative linear random operator on an H^* -algebra A with zero annihilator is probably continuous. In fact, if D is such an operator, then $c(D) \geq \delta(D)$, i.e., the probability of D being stochastically continuous is at least the probability of D being stochastically derivative.*

V. RANDOM DERIVATIONS ON REAL H^* -ALGEBRAS

H^* -algebras over the field of real numbers have been well considered in the literature (cf. [3, 7, 21] for the associative case and [2, 5, 16, 26] for the nonassociative case).

Since every random operator on a real H^* -algebra A can be extended in a natural way to a random operator on the complexification of A (which is trivially an H^* -algebra), preserving the property of being stochastically derivative or probably derivative, it follows, from Corollary 2, the next result.

Corollary 4. *Let D be a probably derivative linear random operator on a real H^* -algebra A , with probability δ of being stochastically derivative. Then every element in the separating subspace for D is valued in the annihilator of A with probability at least δ .*

So stochastically derivative random operators are also stochastically continuous for real H^ -algebras with zero annihilator.*

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