REPRESENTING PROJECTIVE SETS AS UNIONS OF BOREL SETS

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ABSTRACT. We consider a method of representing projective sets by a particular type of union of Borel sets, assuming AD. We prove a generalization of the theorem that a set is Σ_2^1 iff it is the union of ω_1 Borel sets.

The Axiom of Determinacy (AD) is always assumed.

Theorem 1 (Sierpinski, Solovay, Moschovakis [8, 7D.10]). *Let $A \subset \omega^\omega$. A is Σ_2^1 iff A is the union of ω_1 Borel sets.*

The purpose of this paper is to generalize Theorem 1 to higher levels of the projective hierarchy. The pointclass of Σ_2^1 sets is closed under well-ordered unions (Kechris, Solovay, and Steel [6, 2.4.1]), so clearly the higher level projective sets cannot be represented as a well-ordered union of Borel sets. But they will be represented by a special kind of union which characterizes the pointclass Σ_{2n+2}^1 .

Let λ denote a projective ordinal. Let μ_λ denote the supercompact measure on $P_{\omega_1}(\lambda)$ which is defined in Becker [1]. (Woodin [9] has shown that μ_λ is, in fact, the only supercompact measure on $P_{\omega_1}(\lambda)$.) We assume familiarity with the basic facts about μ_λ , all of which can be found in Becker [1]. We also assume familiarity with the theory of projective sets, under AD, as presented in Moschovakis [8]. Let $\mathcal{B}$ denote the class of Borel subsets of ω^ω .

Definition. Let $A \subset \omega^\omega$, and let $F : P_{\omega_1}(\lambda) \rightarrow \mathcal{B}$. F is called a λ -representation of A if for any set $\mathcal{S} \subset P_{\omega_1}(\lambda)$ such that $\mu_\lambda(\mathcal{S}) = 1$, $A = \bigcup_{S \in \mathcal{S}} F(S)$. We say that A is λ -representable if there exists a λ -representation of A .

Proposition 2. *Let $A \subset \omega^\omega$.*

(a) *Let $F : P_{\omega_1}(\lambda) \rightarrow \mathcal{B}$. F is a λ -representation of A iff both of the following properties hold.*

- (i) *For all $S \in P_{\omega_1}(\lambda)$, $F(S) \subset A$.*
- (ii) *For all $x \in \omega^\omega$, if $x \in A$, then for μ_λ -a.e. S , $x \in F(S)$.*
- (b) *A is ω_1 -representable iff A is the union of ω_1 Borel sets.*
- (c) *A is ω -representable iff A is Borel.*

Proof. (b) μ_{ω_1} -a.e. set in $P_{\omega_1}(\omega_1)$ is an initial segment of ω_1 . $\square$

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Thus the λ -representable sets are those that can be obtained by a particular type of union of Borel sets, and this type of union is a generalization of a well-ordered union of length ω_1 , as promised. (By Becker [1, 1.13], for $\lambda \geq \omega_2$, if $\mathcal{S} \subset P_{\omega_1}(\lambda)$ is well-orderable, then $\mu_\lambda(\mathcal{S}) = 0$.) Of course, part (b) of Proposition 2 implies that the Σ_2^1 sets are precisely the δ_1^1 -representable sets. It is this fact which we wish to generalize to higher levels of the projective hierarchy.

Theorem 3. *Let $n \in \omega$. Let $A \subset \omega^\omega$. The following are equivalent.*

- (a) A is Σ_{2n+2}^1 .
- (b) A is δ_{2n+1}^1 -representable.

Theorem 3 and its proof (given below) generalize to scaled pointclasses beyond the projective hierarchy, in a straightforward manner.

In recent years there has been much research on the subject of Borel equivalence relations on ω^ω ; for a survey, see Kechris [5]. One of the major results is the dichotomy theorem of Harrington, Kechris, and Louveau [4]. Theorem 3 has an application to the study, under AD, of this dichotomy for $E \upharpoonright I$, where E is a Borel equivalence relation on ω^ω and $I \subset \omega^\omega$ is an arbitrary E -invariant set. This application will appear in Becker [2].

We next consider two variations on the property of Theorem 3(b): Changing the ordinal and changing the collection of “simple” sets used in the representation.

Let γ_{2n+1} denote the cardinal such that $\delta_{2n+1}^1 = (\gamma_{2n+1})^+$.

Conjecture. Let $n \in \omega$. Let $A \subset \omega^\omega$. If A is Δ_{2n+1}^1 then A is γ_{2n+1} -representable.

Remarks. 1. It can be shown that the converse of this conjecture holds: Every γ_{2n+1} -representable set is Δ_{2n+1}^1 . We do not give the proof in this paper but merely point out that the proof involves work of Kechris and Woodin [7] and Harrington [3].

2. For $n = 0$ the conjecture is trivially true (Proposition 2(c)).

3. By Moschovakis [8, 6E.14], every Δ_{2n+1}^1 set is the one-to-one continuous image of a Π_{2n}^1 set. Since the class of Borel sets is closed under one-to-one continuous images and under Wadge-reducibility, clearly the class of γ_{2n+1} -representable sets also has these two closure properties. Therefore, for $n \geq 1$, one of the following two statements must be true.

- (a) For all $A \subset \omega^\omega$, A is γ_{2n+1} -representable iff A is Δ_{2n+1}^1 .
- (b) For all $A \subset \omega^\omega$, A is γ_{2n+1} -representable iff A is Σ_{2n}^1 iff A is δ_{2n-1}^1 -representable.

Suppose Γ is a pointclass. Define (Γ, λ) -representable by changing $\mathcal{B}$ to $\Gamma \upharpoonright \omega^\omega$ in the definition of λ -representable. (Thus λ -representable means (Borel, λ)-representable.) The fact that Σ_{2n+2}^1 sets are $(\Sigma_1^1, \delta_{2n+1}^1)$ -representable is a trivial corollary of the fact that they are δ_{2n+1}^1 -Suslin. The (a) $\Rightarrow$ (b) direction of Theorem 3 is a strengthening of the above fact: From $(\Sigma_1^1, \delta_{2n+1}^1)$ -representable to (Borel, δ_{2n+1}^1)-representable. It cannot be further strengthened from (Borel, δ_{2n+1}^1)-representable to $(\Sigma_\alpha^0, \delta_{2n+1}^1)$ -representable. In fact, it can be shown, again using Kechris and Woodin [7] and Harrington [3], that for

any projective ordinal λ , for any $\alpha < \omega_1$, and for any $A \subset \omega^\omega$, if A is $(\Sigma_\alpha^0, \lambda)$ -representable, then A is Σ_α^0 ; and similarly for Π_α^0 .

The rest of this paper consists of a proof of Theorem 3. Henceforth we write μ^{2n+1} rather than $\mu_{\delta_{2n+1}^1}$.

Lemma 4. *If A is δ_{2n+1}^1 -representable, then A is Σ_{2n+2}^1 .*

Proof. By Becker [1, 3.1 and 3.2], for any $\lambda < \delta_{2n+1}^1$ and any $G : P_{\omega_1}(\lambda) \rightarrow \mathcal{B}$ (not necessarily representing a set), the set

$$G^* = \{x \in \omega^\omega : \text{For } \mu_\lambda\text{-a.e. } S, x \in G(S)\}$$

is Σ_{2n+2}^1 . And by Becker [1, 2.7], for any $\mathcal{S} \subset P_{\omega_1}(\delta_{2n+1}^1)$, $\mu^{2n+1}(\mathcal{S}) = 1$ iff for a c.u.b. set of $\lambda < \delta_{2n+1}^1$, $\mu_\lambda(\mathcal{S} \cap P_{\omega_1}(\lambda)) = 1$.

Now let $F : P_{\omega_1}(\delta_{2n+1}^1) \rightarrow \mathcal{B}$ be a δ_{2n+1}^1 -representation of A . For $\lambda < \delta_{2n+1}^1$, let $A_\lambda = (F \upharpoonright P_{\omega_1}(\lambda))^*$.

Then each A_λ is Σ_{2n+2}^1 and $A = \bigcup_{\lambda < \delta_{2n+1}^1} A_\lambda$. The Coding Lemma then implies that A is also Σ_{2n+2}^1 . $\square$

Of course, Lemma 4 is the (b) $\Rightarrow$ (a) direction of Theorem 3. Now we take up the (a) $\Rightarrow$ (b) direction.

If S is a set of ordinals, then $\text{ot}(S)$ denotes the ordinal which is the order-type of S , with respect to the usual ordering of ordinals.

Lemma 5. *Let $\lambda < \delta_{2n+1}^1$ and let $G : P_{\omega_1}(\lambda) \rightarrow \omega_1$. For μ^{2n+1} -a.e. $S \in P_{\omega_1}(\delta_{2n+1}^1)$, $\text{ot}(S) \geq G(S \cap \lambda)$.*

Proof. Without loss of generality, $\lambda \geq \gamma_{2n+1}$. Let α be the ordinal such that G represents the α^{th} element of the ultrapower of $\omega_1 \bmod \mu_\lambda$. By Becker [1, 4.7], $\alpha < \delta_{2n+1}^1$. Let $f : \lambda \rightarrow \alpha$ be a surjection, and let $\trianglelefteq$ be the pre-well-ordering of λ induced by f . Let $H : P_{\omega_1}(\lambda) \rightarrow \omega_1$ be the function

$$H(R) = \text{order-type of } (R, \trianglelefteq).$$

As shown in Becker [1, 4.4], H also represents the α^{th} element of the ultrapower. Hence for μ_λ -a.e. R , $H(R) = G(R)$.

Clearly μ^{2n+1} -a.e. $S \in P_{\omega_1}(\delta_{2n+1}^1)$ satisfies the following two conditions.

(a) $H(S \cap \lambda) = G(S \cap \lambda)$.

(b) S is closed under f .

Consider such an S . Since $f[S \cap \lambda] \subset S$,

$$\begin{aligned} \text{ot}(S) &\geq \text{ot}(f[S \cap \lambda]) = \text{order-type of } (S \cap \lambda, \trianglelefteq) \\ &= H(S \cap \lambda) = G(S \cap \lambda). \quad \square \end{aligned}$$

Now fix a Σ_{2n+2}^1 set $A \subset \omega^\omega$. We must show that A is δ_{2n+1}^1 -representable.

Let T be a tree on $\omega \times \delta_{2n+1}^1$ such that $p[T] = A$. For any $S \in P_{\omega_1}(\delta_{2n+1}^1)$ and any $\lambda < \delta_{2n+1}^1$, let $T^{S, \lambda}$ denote $T \cap (\omega \times (S \cap \lambda))^{<\omega}$.

Lemma 6. *There exists a function $B : (P_{\omega_1}(\delta_{2n+1}^1) \times \delta_{2n+1}^1 \times \omega_1) \rightarrow \mathcal{B}$ such that the following three conditions are satisfied.*

(a) For all S, λ , $p[T^{S, \lambda}] = \bigcup_{\alpha < \omega_1} B(S, \lambda, \alpha)$.

(b) For all $S, \lambda, \alpha, \beta$, if $\alpha \leq \beta$, then $B(S, \lambda, \alpha) \subset B(S, \lambda, \beta)$.

(c) For all S, S', λ, α , if $S \cap \lambda = S' \cap \lambda$, then $B(S, \lambda, \alpha) = B(S', \lambda, \alpha)$.

Proof. This is a construction of Lusin and Sierpinski (see Moschovakis [8, 2F.2]). Let $B(S, \lambda, \alpha) =$

$\{x \in \omega^\omega : \text{It is not the case that } T^{S, \lambda}[x] \text{ is well-founded with rank less than or equal to } \alpha, \text{ and for every } \sigma \in T^{S, \lambda}[x], \text{ it is not the case that } T_\sigma^{S, \lambda}[x] \text{ is well-founded with rank exactly } \alpha\}.$

To get a δ_{2n+1}^1 -representation of A , let B satisfy Lemma 6 and define $F : P_{\omega_1}(\delta_{2n+1}^1) \rightarrow \mathcal{B}$ as follows:

$$F(S) = \bigcup_{\lambda \in S} B(S, \lambda, \text{ot}(S)).$$

Clearly for any $S \in P_{\omega_1}(\delta_{2n+1}^1)$, $F(S)$ is Borel and

$$\begin{aligned} F(S) &\subset \bigcup_{\lambda \in S} p[T^{S, \lambda}] \quad (\text{by Lemma 6(a)}) \\ &\subset p[T] = A. \end{aligned}$$

In light of Proposition 2(a), all that remains to be proved is that for any $x \in A$, for μ^{2n+1} -a.e. S , $x \in F(S)$.

Fix $x \in A$. Fix $\lambda < \delta_{2n+1}^1$ such that $x \in p[T \cap (\omega \times \lambda)^{<\omega}]$. Then for μ^{2n+1} -a.e. S , $x \in p[T^{S, \lambda}]$; we now consider only these S 's. By parts (a) and (c) of Lemma 6, there is a function $G : P_{\omega_1}(\lambda) \rightarrow \omega_1$ with the property that for any such S , $x \in B(S, \lambda, G(S \cap \lambda))$. By Lemma 5, for μ^{2n+1} -a.e. S , $\text{ot}(S) \geq G(S \cap \lambda)$, so by Lemma 6(b), for μ^{2n+1} -a.e. S , $x \in B(S, \lambda, \text{ot}(S))$. And for μ^{2n+1} -a.e. S , $\lambda \in S$, hence $B(S, \lambda, \text{ot}(S)) \subset F(S)$.

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