A PYTHAGOREAN INEQUALITY

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ABSTRACT. Let $\{v_1, v_2, v_3, \dots\}$ be a sequence of elements of a Hilbert space, and suppose that (one or both of) the inequalities $d^2 \sum a_i^2 \leq \|\sum a_i v_i\|^2 \leq D^2 \sum a_i^2$ hold for every finite sequence of scalars $\{a_i\}$. If an element v_0 is adjoined to $\{v_i\}$, then the resulting set satisfies (one or both of) $d_0^2 \sum a_i^2 \leq \|\sum a_i v_i\|^2 \leq D_0^2 \sum a_i^2$, where, denoting the norm of v_0 by r and its distance from the closed linear span of the v_i by δ ,

$$d_0^2 = d^2 + \frac{1}{2} \left(r^2 - d^2 - \sqrt{(r^2 + d^2)^2 - 4d^2\delta^2} \right)$$

and

$$D_0^2 = D^2 + \frac{1}{2} \left(r^2 - D^2 + \sqrt{(r^2 + D^2)^2 - 4D^2\delta^2} \right).$$

Both bounds are best possible. If v_0 is in the span of the original set, the expressions above simplify to $d_0 = 0$ and $D_0^2 = D^2 + r^2$. If the original set is a single unit vector v_1 , so $d = D = 1$, and if $v_0 \perp v_1$ is a unit vector so $\delta = 1$, then the above is $(a^2 + b^2) \leq \|av_0 + bv_1\|^2 \leq (a^2 + b^2)$, the Pythagorean Theorem.

Several consequences are deduced. If v_i are unit vectors, $\sum a_i^2 = 1$, and δ_i is the distance from v_i to the span of its predecessors (so that the volume of the parallelotope spanned by the v_i is $V_n = \delta_1 \delta_2 \cdots \delta_n$), the above result is used to show that $\|\sum_{i=0}^n a_i v_i\| \geq V_n / 2^{n/2}$.

1. AN ESTIMATE OF THE NORM OF A LINEAR COMBINATION

This work derives an estimate of the norm of a linear combination in terms of coordinate norms, the resulting inequalities being valid in finite or infinite dimensions. For simplicity of statement and proof two separate theorems are stated. Despite the obvious symmetry, Theorem 1 is likely the more useful, for there are fewer tools which provide nonzero lower bounds for the norm of a sum.

Theorem 1. Let $\{v_i\}$, $i = 1, 2, 3, \dots$, be a sequence of elements of a Hilbert space H , and suppose that the inequality

$$(1) \quad d^2 \sum a_i^2 \leq \left\| \sum a_i v_i \right\|^2$$

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holds for every finite equation of scalars $\{a_i\}$. If an element v_0 is adjoined to $\{v_i\}$, then the resulting set satisfies, for every finite sequence of scalars,

$$(2) \quad d_0^2 \sum a_i^2 \leq \left\| \sum a_i v_i \right\|^2$$

where, denoting the norm of v_0 by r and its distance from the closed linear span of the v_i by δ ,

$$(3) \quad d_0^2 = d^2 + \frac{1}{2} \left(r^2 - d^2 - \sqrt{(r^2 + d^2)^2 - 4d^2\delta^2} \right).$$

If the bound d is sharp, so is d_0 .

Theorem 2. Let $\{v_i\}$, $i = 1, 2, 3, \dots$, be a sequence of elements of a Hilbert space H , and suppose that the inequality

$$(4) \quad \left\| \sum a_i v_i \right\|^2 \leq D^2 \sum a_i^2$$

holds for every finite sequence of scalars $\{a_i\}$. If an element v_0 is adjoined to $\{v_i\}$, then the resulting set satisfies, for every finite sequence of scalars,

$$(5) \quad \left\| \sum a_i v_i \right\|^2 \leq D_0^2 \sum a_i^2$$

where, denoting the norm of v_0 by r and its distance from the closed linear span of the v_i by δ ,

$$(6) \quad D_0^2 = D^2 + \frac{1}{2} \left(r^2 - D^2 + \sqrt{(r^2 + D^2)^2 - 4D^2\delta^2} \right).$$

If the bound D is sharp, so is D_0 .

Proof. We prove only Theorem 1; Theorem 2 is similar.

Let v_0 be given, with its norm denoted by r and its distance from the closed linear span of the original set $\{v_i\}$ denoted by δ .

Let $\{a_0, a_1, \dots, a_n\}$ is a finite sequence of scalars. If $a_0 = 0$, the theorem follows from the observation that d_0 never exceeds d : letting $\delta = r \sin \theta$, observe that the radical becomes $\sqrt{(r^2 + d^2)^2 + 4d^2r^2 \cos^2 \theta}$, and therefore the term in parentheses in (3) is negative.

Therefore, suppose that $a_0 \neq 0$. It will be convenient to use $(\cdot, \cdot)$ to represent the l_2 inner product and $\langle \cdot, \cdot \rangle$ to represent the Hilbert space inner product. Inequality (1) is

$$(7) \quad d^2 \leq \frac{(Ga, a)}{(a, a)}$$

where a represents the sequence of coefficients $\{a_1, a_2, \dots, a_n\}$ and G is the Gram matrix whose (i, j) th entry is $\langle v_i, v_j \rangle$.

Consider a linear combination $a_0 v_0 + a_1 v_1 + \dots + a_n v_n$, with a_0 nonzero. Let V be the span and G be the Gram matrix of the finite set $\{v_1, v_2, \dots, v_n\}$, and let G_0 be the Gram matrix of $\{v_0, v_1, \dots, v_n\}$. G and G_0 are Hermitian and nonnegative definite; for simplicity of exposition suppose that the Hilbert space H is over the field of real scalars so that its inner product is symmetric.

Inequality (1) asserts that the smallest eigenvalue of G is no smaller than d^2 ; it is desired to show that no eigenvalue of G_0 is smaller than d_0^2 .

Eigenvectors of G form an orthonormal basis for $\mathfrak{R}^n$; suppose the eigenvalues of G are $d^2 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, with corresponding eigenvectors $\alpha_1, \alpha_2, \dots, \alpha_n$. Let $e_1, e_2, \dots, e_n$ be the corresponding orthonormal elements of H ; for example, if $\alpha_1 = (c_1, c_2, \dots, c_n)$, then

$$e_1 = (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) / \sqrt{\lambda_1}.$$

The argument that follows is not materially affected by the possibility of repeated eigenvalues; note here that if some $\lambda_j = \lambda_{j+1}$, the corresponding α_j and α_{j+1} are not unique and neither are e_j and e_{j+1} . The reader who ignores the occasional remark concerning repeated eigenvalues will have lost little in what follows. Three claims:

(1) If v_0 is orthogonal to $e_2, e_3, \dots, e_n$, so that $v_0 = (r \cos \theta)e_1 + (r \sin \theta)w$, for w a unit vector in $V^\perp$, then the smallest eigenvalue of G_0 is

$$(8) \quad \begin{aligned} & \lambda_1 + \frac{1}{2} \left(r^2 - \lambda_1 - \sqrt{(r^2 + \lambda_1)^2 - 4\lambda_1 r^2 \sin^2 \theta} \right) \\ &= \lambda_1 + \frac{1}{2} \left(r^2 - \lambda_1 - \sqrt{(r^2 - \lambda_1)^2 + 4\lambda_1 r^2 \cos^2 \theta} \right). \end{aligned}$$

(2) The above is the worst case: if $v_0 = (r \cos \theta)v + (r \sin \theta)w$ for unit vectors $v \in V$ and $w \in V^\perp$, the smallest eigenvalue of G_0 is not smaller than the expression (8) above.

(3) The expression (8) above is an increasing function of λ_1 and of $\sin \theta$ so that establishing the above two statements establishes the theorem: computing with a finite subset of the original $\{v_i\}$ produces values for $r \sin \theta$ and λ_1 which are overestimates of δ and d^2 .

The proof of (2) actually includes the result (1), but the exposition is simpler if the two are separated.

Suppose then that $v_0 = (r \cos \theta)e_1 + (r \sin \theta)w$, for a unit vector w in $V^\perp$. The Gram matrix G_0 is

$$(9) \quad \begin{bmatrix} \langle v_0, v_0 \rangle & \langle v_0, v_1 \rangle & \langle v_0, v_2 \rangle & \cdots & \langle v_0, v_n \rangle \\ \langle v_1, v_0 \rangle & & & & \\ \langle v_2, v_0 \rangle & & & & \\ \langle v_3, v_0 \rangle & & & & \\ \vdots & & & & \\ \langle v_n, v_0 \rangle & & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ G \\ \\ \end{bmatrix}.$$

Notice that $n-1$ of the eigenvectors of G_0 are the eigenvectors $\alpha_2, \alpha_3, \dots, \alpha_n$ of G with a zero adjoined to the first position, with the same eigenvalues; repeated eigenvalues are again repeated with the same multiplicity. For example, if $\alpha_k = (b_1, b_2, \dots, b_n)$, then

$$(10) \quad \begin{bmatrix} \langle v_0, v_0 \rangle & \langle v_0, v_1 \rangle & \langle v_0, v_2 \rangle & \cdots & \langle v_0, v_n \rangle \\ \langle v_1, v_0 \rangle & & & & \\ \langle v_2, v_0 \rangle & & & & \\ \langle v_3, v_0 \rangle & & & & \\ \vdots & & & & \\ \langle v_n, v_0 \rangle & & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ G \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_k b_1 \\ \lambda_k b_2 \\ \vdots \\ \lambda_k b_n \end{bmatrix}$$

where the zero remains in the first position because v_0 is orthogonal to $e_k = b_1v_1 + \cdots + b_nv_n$.

Now suppose that $\alpha_1 = (c_1, c_2, \dots, c_n)$. The two remaining eigenvectors are in the form $v = (x, c_1, c_2, \dots, c_n)$. To determine x , compute the product G_0v directly, observing that, because $\sum_j c_j \langle v_i, v_j \rangle$ is the i th row of $G\alpha_1$,

$$(11) \quad \langle v_i, v_0 \rangle = \left\langle v_i, \frac{r \cos \theta}{\sqrt{\lambda_1}} \sum_j c_j v_j \right\rangle = \frac{r \cos \theta}{\sqrt{\lambda_1}} \sum_j c_j \langle v_i, v_j \rangle = r \cos \theta \sqrt{\lambda_1} c_i.$$

Similarly,

$$(12) \quad \sum_i c_i \langle v_0, v_i \rangle = \left\langle v_0, \sum_i c_i v_i \right\rangle = \langle v_0, \sqrt{\lambda_1} e_1 \rangle = r \cos \theta \sqrt{\lambda_1}$$

so that

$$(13) \quad \begin{bmatrix} \langle v_0, v_0 \rangle & \langle v_0, v_1 \rangle & \langle v_0, v_2 \rangle & \cdots & \langle v_0, v_n \rangle \\ \langle v_1, v_0 \rangle & & & & \\ \langle v_2, v_0 \rangle & & & & \\ \langle v_3, v_0 \rangle & & & & \\ \vdots & & & & \\ \langle v_n, v_0 \rangle & & & & \end{bmatrix} \begin{bmatrix} x \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} r^2 x + r \cos \theta \sqrt{\lambda_1} \\ x r \cos \theta \sqrt{\lambda_1} c_1 + \lambda_1 c_1 \\ x r \cos \theta \sqrt{\lambda_1} c_2 + \lambda_1 c_2 \\ \vdots \\ x r \cos \theta \sqrt{\lambda_1} c_n + \lambda_1 c_n \end{bmatrix}.$$

Eigenvectors correspond to the values of x which are solutions of the quadratic equation

$$x(xr \cos \theta \sqrt{\lambda_1} + \lambda_1) = r^2 x + r \cos \theta \sqrt{\lambda_1},$$

and the eigenvalues are

$$x r \cos \theta \sqrt{\lambda_1} + \lambda_1.$$

Choosing the smaller value of x , which produces an eigenvalue smaller than λ_1 which must therefore be the smallest eigenvalue of G_0 , and replacing $\cos^2 \theta$ with $1 - \sin^2 \theta$ gives (8). It is not difficult to show that both eigenvalues are the same just when $r = \delta_0$, which happens when v_0 is in $V^\perp$, in which case it is easy to see that $\delta_0 = d$.

The second claim above can be approached the same way. Suppose that $v_0 = \sum_{j=1}^n s_j e_j + \delta_0 w$, for a unit vector w in $V^\perp$, where $r \cos \theta = \sqrt{\sum s_i^2}$ and $r \sin \theta = \delta_0$, the distance from v_0 to the span of the finite set $\{v_1, v_2, \dots, v_n\}$. The Gram matrix G_0 still has the form (9), but the expressions (11) and (12) must be generalized. Denoting the coordinates of the α_k by c_{ki} , so that $\alpha_k = (c_{1k}, c_{2k}, \dots, c_{nk})$, and $e_k = \frac{1}{\sqrt{\lambda_k}} \sum_{i=1}^n c_{ik} v_i$, the analog to (11) is, again using

the observation that $\sum_{i=1}^n \langle v_i, v_k \rangle c_{ij}$ is the k th element of $G\alpha_j = \lambda_j \alpha_j$,

$$\begin{aligned}
 \langle v_0, v_k \rangle &= \left\langle \sum_{j=1}^n s_j e_j + \delta_0 w, v_k \right\rangle = \sum_{j=1}^n s_j \langle e_j, v_k \rangle \\
 (14) \quad &= \sum_{j=1}^n s_j \left\langle \frac{1}{\sqrt{\lambda_j}} \sum_{i=1}^n c_{ij} v_i, v_k \right\rangle = \sum_{j=1}^n \frac{s_j}{\sqrt{\lambda_j}} \sum_{i=1}^n \langle v_i, v_k \rangle c_{ij} \\
 &= \sum_{j=1}^n \frac{s_j}{\sqrt{\lambda_j}} \lambda_j c_{kj} = \sum_{j=1}^n s_j \sqrt{\lambda_j} c_{kj}.
 \end{aligned}$$

If any s_j are zero, the corresponding eigenvalues λ_j are undisturbed, with eigenvectors as before with a zero adjoined in the first position. Repeated eigenvalues are repeated with the same multiplicity. If any of the s_j corresponding to a repeated eigenvalue λ_j are nonzero, it is simplest to note that one could choose the original e_j in such a way that one of them is parallel to the projection of v_0 into the associated eigenspace and the others are perpendicular. By that device, one may suppose that each nonzero s_j corresponds to a distinct eigenvalue. The eigenvectors corresponding to nonzero s_j have the form, summing over the nonzero s_j and supposing that there are m such,

$$(15) \quad v = \begin{bmatrix} x \\ \sum_{j=1}^m \beta_j c_{1j} \\ \sum_{j=1}^m \beta_j c_{2j} \\ \vdots \\ \sum_{j=1}^m \beta_j c_{nj} \end{bmatrix} = \begin{bmatrix} x \\ \beta_1 \begin{bmatrix} \alpha_1 \end{bmatrix} + \cdots + \beta_m \begin{bmatrix} \alpha_m \end{bmatrix} \end{bmatrix}.$$

Inserting this into G_0 , using (14) above only in the left column gives

$$(16) \quad \begin{bmatrix} r^2 & \langle v_0, v_1 \rangle & \langle v_0, v_2 \rangle & \cdots & \langle v_0, v_n \rangle \\ \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{1j} & & & & \\ \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{2j} & & & & \\ \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{3j} & & G & & \\ \vdots & & & & \\ \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{nj} & & & & \end{bmatrix}.$$

To compute the first entry in the product $G_0 v$, it is convenient to use the leftmost form in (15) for v . Remembering that the entries of α_k are c_{kj} and using a subscript k outside parentheses to indicate the k th entry of a vector,

$$\begin{aligned}
 (G_0 v)_0 &= r^2 x + \sum_{k=1}^n \langle v_0, v_k \rangle \sum_{j=1}^m \beta_j c_{kj} = r^2 x + \sum_{j=1}^m \beta_j \sum_{k=1}^n c_{kj} \langle v_0, v_k \rangle \\
 (17) \quad &= r^2 x + \sum_{j=1}^m \beta_j \left\langle v_0, \sum_{k=1}^n c_{kj} v_k \right\rangle = r^2 x + \sum_{j=1}^m \beta_j \langle v_0, \sqrt{\lambda_j} e_j \rangle \\
 &= r^2 x + \sum_{j=1}^m \beta_j s_j \sqrt{\lambda_j}.
 \end{aligned}$$

The rightmost form in equation (15) is more convenient for calculating the other entries of the product:

$$\begin{aligned}
 (G_0 v)_k &= x \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{kj} + (\beta_1 G \alpha_1 + \beta_2 G \alpha_2 + \cdots + \beta_m G \alpha_m)_k \\
 (18) \quad &= x \sum_{j=1}^m s_j \sqrt{\lambda_j} c_{kj} + \sum_{j=1}^m \beta_j \lambda_j c_{kj} = \sum_{j=1}^m (x s_j \sqrt{\lambda_j} + \beta_j \lambda_j) c_{kj}
 \end{aligned}$$

so that

$$(19) \quad G_0 v = \begin{bmatrix} \sum_{j=1}^m \beta_j s_j \sqrt{\lambda_j} + x r^2 \\ \sum_{j=1}^m (x s_j \sqrt{\lambda_j} + \beta_j \lambda_j) c_{1j} \\ \sum_{j=1}^m (x s_j \sqrt{\lambda_j} + \beta_j \lambda_j) c_{2j} \\ \vdots \\ \sum_{j=1}^m (x s_j \sqrt{\lambda_j} + \beta_j \lambda_j) c_{nj} \end{bmatrix}.$$

In order for v to be an eigenvector (with eigenvalue λ) it is sufficient that, separately for each j ,

$$(20) \quad \beta_j \lambda = x s_j \sqrt{\lambda_j} + \beta_j \lambda_j$$

and that

$$(21) \quad \sum_{j=1}^m \beta_j s_j \sqrt{\lambda_j} + x r^2 = \lambda x.$$

Dividing (21) by x , in the j th term substituting the value for x provided by solving (20) for x , gives an equation whose $m+1$ solutions are eigenvalues of G_0 :

$$(22) \quad \frac{s_1^2 \lambda_1}{\lambda - \lambda_1} + \frac{s_2^2 \lambda_2}{\lambda - \lambda_2} + \cdots + \frac{s_m^2 \lambda_m}{\lambda - \lambda_m} = \lambda - r^2.$$

Considering the left and right sides of this equation to be functions of λ , one can see the nature of the solutions graphically by sketching each function and looking for their intersections. The left side is a weighted sum of terms of the form

$$(23) \quad f(x) = \frac{a}{x - a};$$

so a sketch of equation (22) is easy (see Figure 1).

FIGURE 1. Eigenvalue intersections

It is straightforward to observe that, among sums having the form of the left side of equation (22) with fixed r and δ , and therefore a fixed value for $\sqrt{r^2 - \delta^2} = \sum s_i^2$, the one with the leftmost intersection with the line $y = x - r^2$ is the one with $s_1 = \sqrt{r^2 - \delta^2} = r \cos \theta$ and all other s_i zero. One might perhaps remark here that in such a case, equation (22) is a quadratic equation for λ , whose smaller root gives the result (8). For example, one may notice that the only intersection of two curves of the form (23) is at $x = 0$ and that (23) is an increasing function of a when $x > 0$, so that for all positive λ ,

$$\frac{\lambda_1}{\lambda - \lambda_1} < \frac{\lambda_2}{\lambda - \lambda_2}.$$

In the region $0 < \lambda < \lambda_1$, all terms in equation (22) have the same sign; after normalizing one has the mean of a collection of functions, the smallest of which is the λ_1 term. The intercept of the line is $(-r^2)$, and the intercept of the sum on the left is $-\sum s_i^2$; the two intercepts are identical if $\delta_0 = 0$, but otherwise the line is beneath the sum at $x = 0$; its intersection with any curve of the form of equation (22) is a minimum when that sum is minimum, which is when all s_j except the first are zero. $\square$

2. APPLICATIONS

The two properties that characterize a Riesz basis for a Hilbert space are completeness and independence, where the required form of independence for a set $\{f_1, f_2, f_3, \dots\}$ is (see [1, 6]) that there exist a nonzero d and finite D such that, for every finite set of scalars $\{a_i\}$,

$$(24) \quad d^2 \sum a_i^2 \leq \left\| \sum a_i f_i \right\|^2 \leq D^2 \sum a_i^2.$$

Theorems 1 and 2 of this work combine to offer an improvement of a classical theorem of Levinson [3], which states that the completeness of a system $\{e^{i\lambda_n t}\}$ in $L^2[-A, A]$ is unaffected if some λ_n is replaced by another number. We add a restriction that the new exponential not be one of the originals, or else the resulting set is of course not independent.

Theorem 3. *The completeness and independence of a system of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2[-A, A]$ are unaffected if some λ_n is replaced by any number distinct from the remaining λ_i .*

Proof. We show only independence; Levinson proved completeness (closure.)

If the given set of exponentials is independent in the sense of (24), then it must be minimal; no element of the set can be a linear combination of the others. If some $\{e^{i\lambda_n t}\}$ is removed from the set, the resulting set is incomplete, or else the original set could not have been independent. The reduced set is independent; let d and D be its upper and lower bounds. A specified $e^{i\lambda_n t}$ cannot be in the closed linear span of the others, since they are an incomplete set [4].

Let $d_0 > 0$ and $D_0 < \infty$ be the values given by Theorem 1 and Theorem 2, and let $\{f_1, f_2, f_3, \dots\}$ be the new set of complex exponentials. Then the inequality (24) with the weakened bounds d_0 and D_0 obtains for any finite set of scalars $\{a_0, a_1, \dots, a_n\}$, showing that the new set is independent.

If the given set of complex exponentials is not independent, and still is not independent when some $e^{i\lambda_n t}$ is removed, it will also fail to be independent when a new element $e^{i\hat{\lambda}_n t}$ is added. If the original set failed to be independent but becomes independent upon the removal of the term $e^{i\lambda_n t}$, that term can be approximated arbitrarily closely by linear combinations of the remaining elements; hence (see [4]) so can any other complex exponential $e^{i\hat{\lambda}_n t}$, so that the modified set also fails to be independent. $\square$

We remark that if the original set of exponentials is independent but not complete, any exponential can be adjoined to it and the resulting set will still be independent.

A set of exponentials which is independent on $L_2[-\pi, \pi]$ in the sense of (24) is an *interpolating sequence*; for any sequence $\{c_i\} \in l_2$ there is a ϕ in L_2 whose Fourier Transform satisfies $\hat{\phi}(\lambda_n) = c_n$. The least-norm interpolating function is the transform of a $\phi \in L_2$ which is in the span of the exponentials; if $\phi = \sum a_n e^{i\lambda_n t}$, $c = \{c_i\}$, $a = \{a_i\}$, and G is the Gram matrix of the exponentials, then $Ga = c$. The square of the norm of ϕ is, taking l_2 inner products,

$$a^T Ga = a^T c = (G^{-1}c)^T c = c^T G^{-1}c,$$

so that $1/d$ is the norm of the mapping from l_2 to L_2 and hence, to the Paley-Wiener space $\mathbf{P}$. The expression for d_0 in Theorem 1 gives the increase in the norm of the interpolating function due to an additional interpolating point.

Theorem 4. Let $v_0, v_1, \dots, v_n$ be unit vectors in a Hilbert space, let θ_j denote the angle between v_j and the subspace spanned by its predecessors $\{v_0, v_1, \dots, v_{j-1}\}$, and let δ_j be the distance from v_j to that subspace. Let $\{a_i\}$ be a set of scalars normalized so that $\sum a_i^2 = 1$. Then

$$(25) \quad (1 - \cos \theta_1)(1 - \cos \theta_2) \cdots (1 - \cos \theta_n) \leq \left\| \sum_{i=0}^n a_i v_i \right\|^2$$

and

$$(26) \quad \left(\frac{\delta_1^2}{2} \right) \left(\frac{\delta_2^2}{2} \right) \cdots \left(\frac{\delta_n^2}{2} \right) \leq \left\| \sum_{i=0}^n a_i v_i \right\|^2,$$

with similar upper bounds.

Proof. A vector v_0 trivially satisfies the hypotheses of the theorem as a singleton set, with $d_0^2 = D_0^2 = 1$. Suppose that $v_0, v_1, \dots, v_n$ are unit vectors. Using (8) with $r = 1$ and $d_0 = 1$ gives a new lower bound $d_1^2 = 1 - \cos \theta_1$. Using (8) again with $r = 1$ and $d_1^2 = 1 - \cos \theta_1$, denoting by θ_2 the angle between v_2 and the span of the set of its predecessors $\{v_0, v_1\}$, one obtains

$$(27) \quad (1 - \cos \theta_1) + \frac{1}{2} \left[\cos \theta_1 - \sqrt{\cos^2 \theta_1 + 4(1 - \cos \theta_1) \cos^2 \theta_2} \right].$$

A simpler but less sharp bound is obtained by overestimating the square root; it is easy to prove that for a and x between 0 and 1,

$$(28) \quad \sqrt{a^2 + 4(1-a)x^2} \leq a + 2(1-a)x.$$

Using this in the bound (27) gives a weaker, but still positive, lower bound $(1 - \cos \theta_1)(1 - \cos \theta_2)$. Repeating and normalizing so $\sum a_i^2 = 1$ to simplify the appearance gives (25).

Replacing $\cos \theta$ with $\sqrt{1 - \delta^2}$ and overestimating the square root with $\sqrt{1 - \delta^2} < 1 - \delta^2/2$ gives (26). The product of the δ_i^2 is the determinant of the Gram matrix and hence, the square of the volume of the parallelepiped whose edges are the v_i (see [2, Eq. 8.8.13]), giving the form of (26) given in the abstract. $\square$

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