LOCAL DERIVATION OF NEST ALGEBRAS

JUN ZHU

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ABSTRACT. We show that every strongly continuous local derivation on a nest algebra is a derivation.

Let H be a separable and complex Hilbert space, and let $B(H)$ denote the set of all linear bounded operators on H . The symbol $x \otimes y$ denotes the rank-1 operator $\langle \cdot, y \rangle x$ for any $x, y \in H$. We identify a closed linear subspace with the corresponding (orthogonal) projection on it. Let $\mathcal{N}$ be a complete nest; then the nest algebra $\text{alg } \mathcal{N}$ is the algebra of operators which leave every member of $\mathcal{N}$ invariant. For any $N \in \mathcal{N}$, we define that $N_- = \bigvee \{L \in \mathcal{N} : L \subsetneq N\}$ and $N_+ = \bigwedge \{L \in \mathcal{N} : N \subsetneq L\}$.

If $\mathcal{A}$ is a Banach algebra, we say that a linear transformation $\varphi: \mathcal{A} \rightarrow \mathcal{A}$ is a local derivation if for every $a \in \mathcal{A}$, there is a derivation $\delta_a: \mathcal{A} \rightarrow \mathcal{A}$, depending on a , such that $\delta_a(a) = \varphi(a)$. Larson and Sourour [4] proved that every local derivation on $B(X)$ is a derivation where X is a Banach space. Kadison [3] obtained that every continuous local derivation on a von Neumann algebra is a derivation. We state our main result as follows.

Theorem 1. *Let φ be a local derivation on a nest algebra which is continuous in the strong operator topology. Then φ is a derivation.*

First, we prove the following two lemmas.

Lemma 1. *Let φ be a strong-operator-topology continuous local derivation on a nest algebra $\text{alg } \mathcal{N}$ and let N belong to $\mathcal{N}$ with $N \neq 0$ and $N_- \neq H$. Then there are B_N and C_N in $B(H)$ with $B_N|_{N^\perp} = 0$ and $C_N|_{N_-} = 0$ such that*

$$(1) \quad \varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f \quad \text{for any } x \in N \text{ and } f \in (N_-)^\perp.$$

Proof. Note that every derivation on a nest algebra is inner [1]. We may find linear maps B_N and C_N on H with $B_N|_{N^\perp} = 0$ and $C_N|_{N_-} = 0$ such that the equality (1) holds, by imitating the proof of Lemmas 1–4 in [4]. So we only need to prove the B_N and $C_N \in B(H)$. For any $x \in N$, $f \in (N_-)^\perp$, and $u, v, w \in H$, we have

$$\begin{aligned} \langle \varphi(x \otimes f)u, v \rangle w &= (w \otimes v) \varphi(x \otimes f)u = (w \otimes v)(x \otimes C_N f + B_N x \otimes f)u \\ &= (\langle x, v \rangle \langle u, C_N f \rangle + \langle u, f \rangle \langle B_N x, v \rangle) w. \end{aligned}$$

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So

$$(2) \quad \langle \varphi(x \otimes f)u, v \rangle = \langle x, v \rangle \langle u, C_N f \rangle + \langle u, f \rangle \langle B_N x, v \rangle.$$

Fix u and f such that $\langle u, f \rangle \neq 0$. For any $\{x_n\}_{n=1}^\infty \subseteq N$ with $x_n \xrightarrow{\|\cdot\|} x$ and $B_N x_n \xrightarrow{\|\cdot\|} y$ ($n \rightarrow \infty$), we have

$$(3) \quad \langle \varphi(x \otimes f)u, v \rangle = \langle x, v \rangle \langle u, C_N f \rangle = \langle u, f \rangle \langle y, v \rangle.$$

By the equalities (2) and (3), we obtain that $\langle v, B_N x - y \rangle = 0$ for any $v \in H$, so $B_N x = y$, thus $B_N|_N$ is a closed operator. It follows that B_N belong to $B(H)$. Similarly, $C_N \in B(H)$. $\square$

Lemma 2. Let N_1 and $N_2 \in \mathcal{N}$ with $N_1 \subseteq N_2$, $N_1 \neq 0$, and $(N_2)_- \neq H$; and let φ , B_{N_i} , and C_{N_i} ($i = 1, 2$) be as in Lemma 1. Then there is a $\lambda(N_1, N_2) \in \mathbb{C}$ such that

(I) $\varphi(x \otimes f) = x \otimes C_{N_1} f + [B_{N_1} + \lambda(N_1, N_2)N_2]x \otimes f$, and

(II) $\varphi(x \otimes f) = x \otimes [C_{N_1} + \lambda(N_1, N_2)(N_1)_-^\perp]f + B_{N_2}x \otimes f$

for any $x \in N_2$, $f \in (N_1)_-^\perp$ with $x \otimes f \in \text{alg}\mathcal{N}$.

Proof. Obviously, (I) implies (II), so we only need to prove the equality (I). Let $x \in N_1$ and $f \in (N_2)_-^\perp$. We have $x \otimes f \in \text{alg}\mathcal{N}$ by Lemma 3.3 in [5]. By Lemma 1, we have

$$(4) \quad x \otimes C_{N_1} f + B_{N_1} x \otimes f = \varphi(x \otimes f) = x \otimes C_{N_2} f + B_{N_2} x \otimes f.$$

So $x \otimes (C_{N_1} - C_{N_2})f = (B_{N_2} - B_{N_1})x \otimes f$; thus there is a $\lambda = \lambda(N_1, N_2) \in \mathbb{C}$ such that $(C_{N_1} - C_{N_2})|_{(N_2)_-^\perp} = -\lambda(N_2)_-^\perp$ and $(B_{N_1} - B_{N_2})|_{N_1} = \lambda N_1$.

Let $x \otimes f$ be in $\text{alg}\mathcal{N}$ where $x \in N_2$ and $f \in (N_1)_-^\perp$. By Lemma 3.3 in [5], there is an $N \in \mathcal{N}$ such that $x \in N$ and $f \in (N_-)_-^\perp$. Without loss of generality, we may assume that $N_1 \subseteq N \subseteq N_2$. Let B_N and $C_N \in B(H)$ and satisfy the equality (1); as above we may find λ_i ($i = 1, 2$) such that

$$\begin{aligned} (C_N - C_{N_2})|_{(N_2)_-^\perp} &= -\lambda_2(N_2)_-^\perp; & (B_N - B_{N_2})|_N &= \lambda_2 N; \\ (C_{N_1} - C_N)|_{(N_-)_-^\perp} &= -\lambda_1(N_-)_-^\perp; & (B_{N_1} - B_N)|_{N_1} &= \lambda_1 N_1. \end{aligned}$$

Thus

$$\begin{aligned} \varphi(x \otimes f) &= x \otimes C_N f + B_N x \otimes f \\ (5) \quad &= x \otimes (C_{N_1} + \lambda_1(N_-)_-^\perp)f + B_N x \otimes f \\ &= x \otimes C_{N_1} f + (B_N + \lambda_1 N)x \otimes f \\ &= x \otimes C_{N_1} f + (B_{N_2} + \lambda_1 N_2 + \lambda_2 N_2)x \otimes f. \end{aligned}$$

We only need to prove that $\lambda(N_1, N_2) = \lambda_1 + \lambda_2$. For any $x \in N_1$ and $f \in N_-^\perp$, using (4) and (5) we see that

$$\begin{aligned} x \otimes C_{N_1} f + (B_{N_2} + \lambda_1 N_2 + \lambda_2 N_2)x \otimes f \\ = x \otimes C_{N_1} f + B_{N_1} x \otimes f = x \otimes C_{N_1} f + (B_{N_1} + \lambda_1 N_2)x \otimes f. \end{aligned}$$

It follows that $\lambda = \lambda(N_1, N_2) = \lambda_1 + \lambda_2$ as desired. $\square$

Proof of Theorem 1. In the following part, we always assume that φ , C_N , B_N , and $\lambda(N_1, N_2)$ satisfy the equalities (1), (I) and (II) where φ is a SOT-continuous local derivation on $\text{alg}\mathcal{N}$. We claim that there are B and C in

$B(H)$ such that

$$(6) \quad \varphi(x \otimes f) = x \otimes Cf + Bx \otimes f \quad \text{for any } x \otimes f \in \text{alg } \mathcal{N}.$$

In fact, if $H_- \neq H$ and $\{O\}_+ \neq \{O\}$, we may choose that $C = C_H$ and $B = \lambda(H, \{O\}_+)\{O\}_+ + B_{\{O\}_+}$. If $H_- \neq H$ and $\{O\}_+ = \{O\}$, then there is a sequence $\{N_n\}_{n=1}^\infty \subseteq \mathcal{N}$ such that $N_n \supsetneq N_{n+1}$ ($n = 0, 1, 2, \dots$), $N_0 = H$, and $\{N_n\}_{n=1}^\infty$ converges to O in the strong operator topology. We define that

$$C_n = C_{N_n} + \lambda(N_n, H)(N_n)^\perp \quad (n = 1, 2, \dots).$$

It is clear that $C_{n+1}|_{(N_n)^\perp} = C_n|_{(N_n)^\perp}$ by Lemma 2. We claim that $\{\|C_n\|: n = 1, 2, \dots\}$ be bounded. Suppose that $\sup\{\|C_n\|: n = 1, 2, \dots\} = \infty$; then there are sequences $\{n_k\}_{k=1}^\infty \subseteq \mathbb{N}$ and $f_k \in (N_{n_k})^\perp$ with $\|f_k\| = (1/2)^k$ ($k = 1, 2, \dots$) such that $\|C_{n_k}f_k\| > k$ ($k = 1, 2, \dots$). Let $\{x_k\}_{k=1}^\infty$ be a unit vector sequence with $x_k \in N_{n_k}$ and $x_i \perp x_j$ ($i \neq j$). We define an operator

$$T = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \otimes f_i \in \text{Alg } \mathcal{N}.$$

Thus we have

$$\begin{aligned} \left\| \varphi(T) \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|} \right\| &= \left\| \lim_{n \rightarrow \infty} \sum_{i=1}^n \varphi(x_i \otimes f_i) \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n (x_i \otimes C_{n_i}f_i + B_H x_i \otimes f_i) \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|} \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n \left\langle \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|}, C_{n_i}f_i \right\rangle x_i + \left\langle \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|}, f_i \right\rangle B_H x_i \right\| \\ &\geq \lim_{n \rightarrow \infty} \left(\left\| \sum_{i=1}^n \left\langle \frac{C_{n_k}f_k}{\|C_{n_k}f_k\|}, C_{n_i}f_i \right\rangle x_i \right\| - \sum_{i=1}^n \frac{1}{2^i} \|B_H\| \right) \\ &\geq \|C_{n_k}f_k\| - \|B_H\| > k - \|B_H\|, \end{aligned}$$

contradicting the fact that $\|\varphi(T)\| < \infty$.

It is clear that $C_{n+1}|_{(N_n)^\perp} = C_n|_{(N_n)^\perp}$ by Lemma 2, so we can define an operator C by $Cx = \lim_{n \rightarrow \infty} C_n x$ for every $x \in H$ and take $B = B_H$. It follows that the operators B and C satisfy the equality (6). The case $H_- = H$ can be proved by a similar argument. It follows that the equation (6) holds for any rank-1 operator $x \otimes f \in \text{alg } \mathcal{N}$. By Theorems 1 and 2 of [2], the linear combination of the set $\{x \otimes f: x \otimes f \in \text{alg } \mathcal{N}\}$ is dense in the strong operator topology, so we have

$$(7) \quad \varphi(T) = TC^* + BT \quad \text{for any } T \in \text{alg } \mathcal{N}.$$

In the equality (7), taking $T = I$, we get that $C^* + B = 0$. Thus $C^* = -B$. So φ is a derivation. $\square$

REFERENCES

1. E. Christensen, *Derivations of nest algebras*, Math. Ann. **229** (1977), 155–161.
2. J. A. Erdos, *Operators of finite rank in nest algebras*, J. London Math. Soc. **43** (1968), 391–397.
3. R. Kadison, *Local derivations*, preprint.

4. D. R. Larson and A. R. Sourour, *Local derivation and local automorphisms of $B(X)$* , Operator Algebras and Applications, Proc. Sympos. Pure Math., vol. 51, Part 2, Amer. Math. Soc., Providence, RI, 1990, pp. 187–194.
5. J. R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. (3) **15** (1965), 61–83.

DEPARTMENT OF MATHEMATICS, HUBEI INSTITUTE FOR NATIONALITIES, ENSHI 445000, PEOPLES
REPUBLIC OF CHINA