A COUNTEREXAMPLE TO ACCESS THEOREMS FOR C^∞ FUNCTIONS

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ABSTRACT. We construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with the following properties:

- (1) f is of class ∞ .
- (2) If $m \in \mathbb{R}^2$, then $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ exists such that $\gamma(0) = m$, γ is continuous, and $f \circ \gamma$ is strictly increasing on $[0, 1]$.
- (3) If $\sigma : [0, 1] \rightarrow \mathbb{R}^2$ is continuous and $f \circ \sigma$ is nondecreasing on $[0, 1]$, then $\sup\{|\sigma(s)| : 0 \leq s < 1\} < \infty$.

INTRODUCTION

A few years ago, Professor W. K. Hayman proposed the following statement concerning the topology of analytic¹ functions:

Access Theorem for Analytic Functions. *Suppose that $\mathcal{M}$ is an analytic manifold, $m \in \mathcal{M}$, $f : \mathcal{M} \rightarrow \mathbb{R}$, and f is analytic. Then at least one of the following three statements is true:*

- (1) m is a local maximum of f .
- (2) A mapping $\sigma : [0, 1] \rightarrow \mathcal{M}$ exists such that σ is a strict f path, $\sigma(0) = m$, and $\sigma(1)$ is a local maximum of f .
- (3) A mapping $\sigma : [0, 1] \rightarrow \mathcal{M}$ exists such that σ is a strict f path, $\sigma(0) = m$, and $\text{Cluster}(\sigma) = \emptyset$.

Previously, similar theorems had been established for use in potential theory, especially for studying the asymptotic behavior of subharmonic functions. In this area, the theorems of Hornblower, Thomas, and Hayman ([H-T], [H]) are of particular importance, and natural questions suggested by those results instigated the development of the access theorem for analytic functions. The proof of the access theorem, which appears in [O], uses developments of the fundamental work of Bruhat and Cartan in singularity theory ([B-C]).

At the outset, it is clear that any generalization of the access theorem to the class C^∞ must explicitly impose the following restriction on the local behavior

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¹The term *analytic* means *real-analytic*. All terminology is discussed in §2.

of the function f :

(*) *Every point in $\text{dmn } f$ is either a local maximum of f or an $f \uparrow$ point.*

Indeed, condition (*) holds whenever f satisfies the conclusion of the access theorem, but not every function of class ∞ satisfies condition (*). (For an example, define $f: \mathbb{R} \rightarrow \mathbb{R}$ by setting $f(0) = 0$ and $f(x) \equiv e^{-1/x^2} \sin(1/x)$ whenever $x \in \mathbb{R} \setminus \{0\}$, and note that f is of class ∞ , 0 is not a local maximum of f , and 0 is not an $f \uparrow$ point.) However, the problem of determining which restriction (if any) actually supports a generalization of the access theorem appears to be complicated, and the following construction, which frames this problem to an extent, shows that any such restriction must be *properly stronger* than condition (*).

Theorem 1.1. *A function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ exists with the following properties:*

- (1) *f is of class ∞ .*
- (2) *Every point of $\mathbb{R}^2$ is an $f \uparrow$ point (in particular, f has no local maxima).*
- (3) *If $\sigma: [0, 1) \rightarrow \mathbb{R}^2$, σ is a weak f path, $n \in \mathbb{N}$, and $|\sigma(0)| < e^n$, then $|\sigma(t)| < e^n$ whenever $t \in [0, 1)$ (and, hence, $\text{Cluster}(\sigma) \neq \emptyset$).*

An examination of the proof of Theorem 1.1 shows that the critical property (3) of f derives from the structure of the set of $f \downarrow$ points, which is not connected in $\mathbb{R}^2$. This relationship suggests the following specific question, which I cannot resolve: Suppose that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, f is of class ∞ , every $x \in \mathbb{R}^2$ is an $f \uparrow$ point, and every $x \in \mathbb{R}^2$ is an $f \downarrow$ point. Under these conditions, must there exist a mapping $\sigma: [0, 1) \rightarrow \mathbb{R}^2$ such that σ is a strict f path and $\sup\{|\sigma(t)| : t \in [0, 1)\} = \infty$?

2. NOTATION AND TERMINOLOGY

We set $\mathbb{N} \equiv \{1, 2, 3, \dots\}$ and follow [Fed] (pp. 669-671) for all other notation in set theory and topology. The symbols D^k (k th differential) and $D^{(\alpha_1, \alpha_2)}$ (mixed partial derivatives) are also defined in [Fed].

We say that $\mathcal{M}$ is an analytic manifold and $\mathcal{N}$ is manifold of class ∞ if $\mathcal{M}$ and $\mathcal{N}$ are second countable Hausdorff spaces that meet the conditions of [St] (pp. 32-33). In the case that f and g are real-valued functions, $\text{dmn } f$ is an open subset of an analytic manifold, and $\text{dmn } g$ is an open subset of a manifold of class ∞ , we say that f is an analytic function and g is a function of class ∞ if f and g meet the conditions of [St] (pp. 32-33). In the case that $\mathcal{N}$ is a manifold of class ∞ , f is a real-valued function, and $S \subset \text{dmn } f \subset \mathcal{N}$, we say that f is of class ∞ on S if there exists a real-valued function g , defined on an open subset of $\mathcal{N}$, such that g is a function of class ∞ , $S \subset \text{dmn } g$, and $f|_S = g|_S$ (the restrictions of f and g to S). In the case that $\mathcal{N}$ is also an analytic manifold and g is also an analytic function, we say that f is analytic on S .

We use the following terminology when (S, τ) is a topological space, $\text{dmn } f \subset S$, and $\text{im } f \subset \mathbb{R}$:

- (1) We say that s_0 is a local maximum of f if an open set U exists such that $s_0 \in U \cap \text{dmn } f$ and $f(u) \leq f(s_0)$ for all $u \in U \cap \text{dmn } f$.
- (2) We say that σ is a weak f path if $\sigma: I \rightarrow S$, $I = [0, 1)$ or $I = [0, 1]$, σ is continuous, and $f \circ \sigma(t_1) \leq f \circ \sigma(t_2)$ whenever $t_1, t_2 \in I$ and $t_1 \leq t_2$.
- (3) We

say that σ is a strict f path if σ is a weak f path and $f \circ \sigma$ is one to one. (4) We say that s_0 is an $f \uparrow$ point if $\sigma : [0, 1) \rightarrow S$ exists such that σ is a strict f path and $\sigma(0) = s_0$. (5) We say that s_0 is an $f \downarrow$ point if s_0 is a $(-f) \uparrow$ point. (6) If $\gamma : [0, 1) \rightarrow S$, we define $\text{Cluster}(\gamma) \equiv \bigcap_{s \in [0, 1)} \text{Clos} \{ \gamma(t) : t \in [s, 1) \}$.

We define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by setting $\phi(r) \equiv 0$ whenever $r \in (-\infty, 0]$,

$$\phi(r) \equiv c^{-1} \int_0^r \exp\{-x^{-1}(1-x)^{-1}\} dx,$$

whenever $r \in (0, 1]$ (here, $c \equiv \int_0^1 \exp\{-x^{-1}(1-x)^{-1}\} dx$), and $\phi(r) \equiv 1$ whenever $r \in [1, \infty)$. The following five properties of ϕ , which we display for reference, are easily verified:

- (1) $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and ϕ is of class ∞ .
- (2) $\phi(r) = 0$ whenever $r \leq 0$.
- (3) ϕ is strictly increasing on $[0, 1]$.
- (4) $\phi(r) = 1$ whenever $r \geq 1$.
- (5) Whenever $k, n \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$, we have

$$\lim_{r \rightarrow 0} (D^k \phi(r)) r^{-n} = 0 \quad \text{and} \quad \lim_{r \rightarrow 1} (D^l \phi(r)) (1-r)^{-n} = 0.$$

3. LEMMAS

Lemma 3.5 is the main step in our construction. There, we build a function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that every point in $\mathbb{R} \times [0, 1)$ is a $G \uparrow$ point and every weak G path connecting the line $y = 0$ to the line $y = 1$ is forced to *oscillate* by a certain fixed amount. The function required in Theorem 1.1 is constructed by piecing together functions of this basic type.

Lemma 3.1. Suppose that Ω is an open subset of $\mathbb{R}^2$ and $\mathbb{R} \times \{0\} \subset \Omega$. Define $\Omega^+ \equiv \{(x, y) \in \Omega : y > 0\}$ and $\Omega^- \equiv \{(x, y) \in \Omega : y < 0\}$. Suppose that $A : \Omega \rightarrow \mathbb{R}$, A is continuous, $A|_{\Omega^+ \cup \Omega^-}$ is of class ∞ , and $D^{(p, q)}(A|_{\Omega^+ \cup \Omega^-})$ has a continuous extension to Ω whenever $p, q \in \mathbb{N} \cup \{0\}$. Then A is of class ∞ .

Proof. Let θ_1 and θ_2 denote, respectively, the continuous extensions to Ω of the functions $D^{(1, 0)}(A|_{\Omega^- \cup \Omega^+})$ and $D^{(0, 1)}(A|_{\Omega^- \cup \Omega^+})$. Then

$$A(x_2, y) - A(x_1, y) = \int_{x_1}^{x_2} \theta_1(r, y) dr,$$

$$A(x, y_2) - A(x, y_1) = \int_{y_1}^{y_2} \theta_2(x, r) dr$$

whenever the closed segment connecting (x_1, y) to (x_2, y) lies in Ω and the closed segment connecting (x, y_1) to (x, y_2) also lies in Ω . Thus, A is continuously differentiable. The proof is completed by induction.

Lemma 3.2. Suppose that $A : \mathbb{R}^2 \rightarrow \mathbb{R}$, $A(x, y) = 0$ whenever $(x, y) \in \mathbb{R} \times (-\infty, 0]$, $A(x, y) = 1$ whenever $(x, y) \in \mathbb{R} \times [1, \infty)$, and A is of class ∞ . Define $a_n \equiv 1 - (1/2)^{n-1}$ whenever $n \in \mathbb{N}$. Define $B : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting

$$B(x, y) \equiv \sum_{n=1}^{\infty} (\phi(a_{n+1}) - \phi(a_n)) A\left(x, \frac{y - a_n}{a_{n+1} - a_n}\right)$$

whenever $(x, y) \in \mathbb{R}^2$. Then B is of class ∞ .

Proof. B has the following four properties: (1) B is continuous. (2) $B|\mathbb{R} \times (-\infty, 1)$ is of class ∞ . (3) If $a, b \in \mathbb{R}$, $p, q \in \mathbb{N} \cup \{0\}$, $a < b$, and $p+q > 0$, then

$$\lim_{n \uparrow \infty} \max\{|D^{(p,q)}B(x, y)| : x \in [a, b], a_n \leq y \leq a_{n+1}\} = 0$$

(use property (5) of ϕ with $l = 1$ and $n = q - 1$). (4) $B|\mathbb{R} \times (1, \infty)$ is identically 1. Thus, by Lemma 3.1, B is of class ∞ .

Lemma 3.3. Suppose that A is of class ∞ on $\mathbb{R} \times [0, 1]$. Define the function $B : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting $B(x, y) \equiv A(x, \phi(y))$ whenever $(x, y) \in \mathbb{R}^2$. Then B is of class ∞ .

Proof. A is the restriction to $\mathbb{R} \times [0, 1]$ of a function of class ∞ (whose domain is therefore open). Thus, B is the composition of two functions of class ∞ .

Lemma 3.4. Suppose that $h : \mathbb{R} \rightarrow \mathbb{R}$, h is of class ∞ , and $h(x+1) = h(x)$ whenever $x \in \mathbb{R}$. Then $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ exists with the following properties:

- (1) H is of class ∞ .
- (2) $H|\mathbb{R} \times (0, 1)$ is harmonic.
- (3) $H(x, 0) = 0$ and $H(x, 1) = h(x)$ whenever $x \in \mathbb{R}$.
- (4) $H(x+1, y) = H(x, y)$ whenever $(x, y) \in \mathbb{R}^2$.

Proof. We use the theory of the Dirichlet problem for harmonic functions on $\mathbb{C} \equiv \mathbb{R}^2$. Set $\Omega \equiv \{z \in \mathbb{R}^2 : 1 < |z| < e^{2\pi}\}$. Then $\lambda : \text{Clos } \Omega \rightarrow \mathbb{R}$ exists such that λ is continuous, $\lambda|_{\Omega}$ is harmonic, $\lambda(z) = 0$ whenever $|z| = 1$, and $\lambda(e^{2\pi-2\pi i x}) = h(x)$ whenever $x \in \mathbb{R}$ ([B-G], pp. 394 and 396).

Moreover, λ is of class ∞ on $\text{Clos } \Omega$. For a brief proof of this, let μ denote the restriction of λ to the circle $\{z \in \mathbb{R}^2 : |z| = e^{2\pi}\}$ and let $P[\mu]$ denote the Poisson extension of μ to the closed disk $\overline{D} \equiv \{z \in \mathbb{R}^2 : |z| \leq e^{2\pi}\}$ ([B-G], p. 310). As μ is of class ∞ on $\text{Bdry } \overline{D}$, it follows that $P[\mu]$ is of class ∞ on $\overline{D}$ ([B-G], p. 430). By the reflection principle, $\lambda - P[\mu]$ is analytic on $\{z \in \mathbb{R}^2 : 1 < |z| \leq e^{2\pi}\}$. Thus, λ is of class ∞ on $\{z \in \mathbb{R}^2 : 1 < |z| \leq e^{2\pi}\}$. By the reflection principle, λ is analytic on $\{z \in \mathbb{R}^2 : 1 \leq |z| < e^{2\pi}\}$. It follows that λ is of class ∞ on $\text{Clos } \Omega$.

Therefore, $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ exists such that Λ is of class ∞ and $\Lambda(z) = \lambda(z)$ whenever $z \in \text{Clos } \Omega$ (multiply λ by a cutoff function of class ∞). Now define $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting $H(x, y) \equiv \Lambda(e^{2\pi y - 2\pi i x})$ whenever $(x, y) \in \mathbb{R}^2 \equiv \mathbb{C}$ and verify that H has the required properties.

Lemma 3.5. $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\epsilon \in (0, \infty)$ exist with the following properties:

- (1) G is of class ∞ .
- (2) $G(x, y) = 0$ whenever $(x, y) \in \mathbb{R} \times (-\infty, 0]$.
- (3) $G(x, y) = 1$ whenever $(x, y) \in \mathbb{R} \times [1, \infty)$.
- (4) $-1 < G(x, y) < 1$ whenever $x \in \mathbb{R}$ and $0 < y < 1$.
- (5) $G(x+1, y) = G(x, y)$ whenever $(x, y) \in \mathbb{R}^2$.
- (6) If $(x, y) \in \mathbb{R} \times [0, 1)$ and $x \in \mathbb{R}$, then (x, y) is a $G \uparrow$ point.
- (7) If $\gamma : [0, 1] \rightarrow \mathbb{R}^2$, γ is a weak G path, $\gamma = (\gamma_1, \gamma_2)$, $\gamma_2(0) = 0$, and $\gamma_2(1) = 1$, then $s, t \in [0, 1]$ exist such that $|\gamma_1(s) - \gamma_1(t)| \geq \epsilon$.

Proof. For each $\delta \in (0, 1/4)$, define $g_\delta: \mathbb{R} \rightarrow \mathbb{R}$ by the following three rules:

- (i) $g_\delta(r) = \frac{3}{2}\phi(\frac{r-\delta}{\delta}) - 1$ whenever $r \in [0, 1/2]$;
- (ii) $g_\delta(r) = g_\delta(1-r)$ whenever $r \in [0, 1]$;
- (iii) $g_\delta(r) = g_\delta(r+1)$ whenever $r \in \mathbb{R}$.

Then g_δ is of class ∞ whenever $\delta \in (0, 1/4)$. Let H_δ denote the function constructed in Lemma 3.4 with $h = g_\delta$. Then $\epsilon \in (0, 1/16)$ exists (suitably small) such that $H_{4\epsilon}(x, y) > 0$ whenever $0 < y < 1/2$. Set $H \equiv H_{4\epsilon}$. Then every point of $\mathbb{R} \times [0, 1)$ is an $H \uparrow$ point (as we deduce from the reflection principle and other elementary properties of harmonic functions).

Next, we define $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$F(x, y) \equiv \begin{cases} 0, & (x, y) \in \mathbb{R} \times (-\infty, 0], \\ H(x, \phi(2y)), & (x, y) \in \mathbb{R} \times [0, 1/2), \\ H(x, 1)(1 - \phi(2y - 1)) + \phi(2y - 1), & (x, y) \in \mathbb{R} \times [1/2, 1), \\ 1, & (x, y) \in \mathbb{R} \times [1, \infty). \end{cases}$$

Then F is of class ∞ (as we deduce from Lemma 3.3 and the properties of ϕ), and every $(x, y) \in \mathbb{R} \times [0, 1)$ is an $F \uparrow$ point. Moreover, if $\sigma: [0, 1] \rightarrow \mathbb{R}^2$, σ is a weak F path, $\sigma = (\sigma_1, \sigma_2)$, $\sigma_2(0) = 0$, $\sigma_2(1) = 1$, and $k \in \mathbb{Z}$, then $|\sigma_1(s) - k| \geq 4\epsilon$ for at least one $s \in [0, 1]$ (as we deduce from the properties of $g_{4\epsilon}$).

Finally, to complete the construction of G , we *stagger* a succession of the barriers constructed above. Thus, choose $N \in \mathbb{N}$ with $N > 1/\epsilon$ and define

$$G(x, y) \equiv \frac{1}{N} \sum_{n=0}^{N-1} F\left(x - \frac{n}{N}, Ny - n\right)$$

whenever $(x, y) \in \mathbb{R}^2$.

Direct verification shows that G has properties (1) through (6) of Lemma 3.5. To prove that G has property (7), take γ as indicated in property (7) and choose $k_0 \in \mathbb{Z}$ and $n \in \{0, 1, \dots, N-1\}$ so that

$$\left| \gamma_1(0) - k_0 - \frac{n}{N} \right| < \epsilon :$$

We prove that a deviation of the required type occurs between $\gamma(0)$ and some point lying on the part of γ between the lines $y = n/N$ and $y = (n+1)/N$. Select $t_1, t_2 \in [0, 1]$ so that $\gamma_2(t_1) = \frac{n}{N}$, $\gamma_2(t_2) = \frac{n+1}{N}$, and so that $\frac{n}{N} \leq \gamma_2(t) \leq \frac{n+1}{N}$ whenever $t \in [t_1, t_2]$. Define $\sigma: [0, 1] \rightarrow \mathbb{R}^2$ as follows: $\sigma \equiv (\sigma_1, \sigma_2)$,

$$\sigma_1(s) \equiv \gamma_1((1-s)t_1 + st_2) - \frac{n}{N} \quad \text{and} \quad \sigma_2(s) \equiv N\gamma_2((1-s)t_1 + st_2) - n$$

whenever $s \in [0, 1]$. The map

$$s \rightarrow F(\sigma(s)) = NG(\gamma((1-s)t_1 + st_2)) - n$$

is non-decreasing on $[0, 1]$ (because γ is a weak G path). Thus, σ is a weak F path, $\sigma_2(0) = 0$, and $\sigma_2(1) = 1$. Recalling the main property of F , we see that $s_0 \in [0, 1]$ exists such that $|\sigma_1(s_0) - k_0| > 4\epsilon$. For the numbers s and t

required by property (7), choose $s \equiv (1 - s_0)t_1 + s_0t_2$ and $t \equiv 0$. Then

$$\begin{aligned} |\gamma_1(s) - \gamma_1(t)| &= |\gamma_1((1 - s_0)t_1 + s_0t_2) - \gamma_1(0)| \\ &\geq |\gamma_1((1 - s_0)t_1 + s_0t_2) - \frac{n}{N} - k_0| - |\gamma_1(0) - \frac{n}{N} - k_0| \\ &\geq |\sigma_1(s_0) - k_0| - \epsilon \geq 4\epsilon - \epsilon > \epsilon, \end{aligned}$$

as required.

Lemma 3.6. $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ exists with the following properties:

- (1) F is of class ∞ .
- (2) $F(x, y) = 0$ whenever $(x, y) \in \mathbb{R} \times (-\infty, 0]$.
- (3) $F(x, y) = 1$ whenever $(x, y) \in \mathbb{R} \times [1, \infty)$.
- (4) $-1 < F(x, y) < 1$ whenever $(x, y) \in \mathbb{R} \times (0, 1)$.
- (5) $F(x + 1, y) = F(x, y)$ whenever $(x, y) \in \mathbb{R}^2$.
- (6) If $(x, y) \in \mathbb{R} \times [0, 1)$, then (x, y) is a $F \uparrow$ point.
- (7) If $\gamma : [0, 1) \rightarrow \mathbb{R}^2$, γ is a weak F path, $\gamma = (\gamma_1, \gamma_2)$, and $\gamma_2(0) < 1$, then $\gamma_2(t) < 1$ whenever $t \in [0, 1)$.

Proof. Choose $\epsilon > 0$ and G as in Lemma 3.5. Define $a_n \equiv 1 - (1/2)^{n-1}$ whenever $n \in \mathbb{N}$, and set

$$F(x, y) \equiv \sum_{n=1}^{\infty} (\phi(a_{n+1}) - \phi(a_n)) G\left(x, \frac{y - a_n}{a_{n+1} - a_n}\right).$$

whenever $(x, y) \in \mathbb{R}^2$. We use Lemma 3.2 to verify property (1), and properties (2) through (6) follow directly from the properties of G . Thus, to complete the proof, suppose that γ meets the hypotheses of property (7) and $t \in [0, 1)$. If $n \in \mathbb{N}$, $0 \leq \alpha < \beta < t$, $\gamma_2(\alpha) = a_n$, and $\gamma_2(\beta) = a_{n+1}$, then

$$\sup\{|\gamma_1(r) - \gamma_1(s)| : \alpha \leq r, s \leq \beta\} \geq \epsilon$$

(by the main property of G). As γ is continuous on $[0, t]$, we conclude that only a finite number of the a_n 's are in the set $\{\gamma_2(s) : s \in [0, t]\}$. As $\gamma_2(0) < 1$, we conclude that $\gamma_2(t) < 1$.

Lemma 3.7. $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ exists with the following properties:

- (1) G is of class ∞ .
- (2) $G(x, y) = \phi(e^y)$ whenever $(x, y) \in \mathbb{R} \times (-\infty, 0]$.
- (3) $G(x + 2\pi, y) = G(x, y)$ whenever $(x, y) \in \mathbb{R}^2$.
- (4) Every $(x, y) \in \mathbb{R}^2$ is a $G \uparrow$ point.
- (5) If $\gamma : [0, 1) \rightarrow \mathbb{R}^2$, γ is a weak G path, $\gamma = (\gamma_1, \gamma_2)$, $n \in \mathbb{N}$, and $\gamma_2(0) < n$, then $\gamma_2(t) < n$ whenever $t \in [0, 1)$.

Proof. Choose F as in Lemma 3.6, and define $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$G(x, y) \equiv \begin{cases} \phi(e^y), & (x, y) \in \mathbb{R} \times (-\infty, 0], \\ 1 + \sum_{n=0}^{\infty} F\left(\frac{x}{2\pi}, y - n\right), & (x, y) \in \mathbb{R} \times (0, \infty). \end{cases}$$

That G has the required properties now follows from Lemma 3.1 and the properties of F and ϕ .

4. PROOF OF THEOREM 1.1

Define $p: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by setting $p(z) \equiv e^{-iz}$ whenever $z \in \mathbb{R}^2 \equiv \mathbb{C}$. Choose G as specified in Lemma 3.7, and (with attention to property (3) of G) define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting $f(0) = 0$ and

$$f(p(z)) = f(e^{-iz}) = G(z)$$

whenever $z \in \mathbb{R}^2$. We now prove that f has the properties required in Theorem 1.1.

As $Dp(z)$ is nonsingular whenever $z \in \mathbb{R}^2$, we conclude that the restriction of f to $\mathbb{R}^2 \sim \{0\}$ is of class ∞ and that every $w \in \mathbb{R}^2 \sim \{0\}$ is an $f \uparrow$ point. Moreover, the restriction of f to a neighborhood of 0 is of class ∞ and 0 is an $f \uparrow$ point (as we deduce from the equation $f(w) = \phi(e^{\log |w|}) = \phi(|w|)$, which holds whenever $0 < |w| < 1$). Therefore, f is of class ∞ and every $w \in \mathbb{R}^2$ is an $f \uparrow$ point.

Finally, suppose that $n \in \mathbb{N}$, σ is a weak f path, $t_0 \in [0, 1)$, and $|\sigma(0)| < e^n$; to complete the proof, we must prove that $|\sigma(t_0)| < e^n$. For that purpose, we may assume that $|\sigma(s)| \neq 0$ for all $s \in [0, 1)$ (replace σ with any part of σ not passing through 0). Thus, a continuous mapping (a lifting) $\gamma: [0, 1) \rightarrow \mathbb{R}^2$ exists such that $p(\gamma(s)) = \sigma(s)$ whenever $s \in [0, 1)$ ([B-G], p. 50). Set $\gamma \equiv (\gamma_1, \gamma_2)$. Then $\gamma_2(0) < n$ (because $|\sigma(0)| = e^{\gamma_2(0)} < e^n$) and γ is a weak G path (because σ is a weak f path and $G(\gamma(s)) = f(p(\gamma(s))) = f(\sigma(s))$ whenever $s \in [0, 1)$). Therefore, $\gamma_2(t_0) < n$ (by property (5) of G in Lemma 3.7) and $|\sigma(t_0)| = e^{\gamma_2(t_0)} < e^n$.

REFERENCES

- [B-C] F. Bruhat and H. Cartan, *Sur la structure des sous-ensembles analytiques réels*, C. R. Acad. Sci. Paris Ser. I Math. **244** (1957), 988–990.
- [B-G] C. Berenstein and R. Gay, *Complex variables*, Springer-Verlag, New York, 1991.
- [Fed] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.
- [H] W. K. Hayman, *Subharmonic functions*, Vol. 2, Academic Press, New York, 1989, p. 780.
- [H-T] R. J. M. Hornblower and E. S. Thomas, *The access theorem for subharmonic functions*, Trans. Amer. Math. Soc. **172** (1972), 287–297.
- [O] M. Ortel, *An access theorem for analytic functions*, Trans. Amer. Math. Soc. (to appear).
- [St] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

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