MENGER MANIFOLDS HOMEOMORPHIC TO THEIR n -HOMOTOPY KERNELS

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ABSTRACT. We give a necessary and sufficient condition that an $(n+1)$ -dimensional Menger manifold (μ^{n+1} -manifold) is homeomorphic to its n -homotopy kernel. Such a μ^{n+1} -manifold is called a μ_∞^{n+1} -manifold. We also prove the following results:

- (1) Each homeomorphism between two Z -sets in a μ_∞^{n+1} -manifold M extends to an ambient homeomorphism of M onto itself if it is n -homotopic to id in M .
- (2) An n -homotopy equivalence between two μ_∞^{n+1} -manifolds is n -homotopic to a homeomorphism.
- (3) Each map from a μ_∞^{n+1} -manifold into a μ^{n+1} -manifold is n -homotopic to an open embedding.

INTRODUCTION

All spaces considered in this paper are assumed to be locally compact separable metrizable and maps are continuous. In [Be], Bestvina introduced Menger manifolds and established the characterization theorem of such manifolds. An $(n+1)$ -dimensional Menger manifold is a topological manifold modeled on the $(n+1)$ -dimensional universal Menger compactum μ^{n+1} , which is also called a μ^{n+1} -manifold. In [Ch3], Chigogidze introduced the notion of the n -homotopy kernel of a μ^{n+1} -manifold and proved the following classification theorem for μ^{n+1} -manifolds: Two μ^{n+1} -manifolds have the same n -homotopy type if and only if their n -homotopy kernels are homeomorphic. There are close relations between Hilbert cube manifold (Q -manifold) theory, and Menger manifold theory, and the n -homotopy kernel of a μ^{n+1} -manifold plays the role of the product $X \times [0, 1)$ of a Q -manifold X with $[0, 1)$. It is said that X is $[0, 1)$ -stable if it is homeomorphic to $(\cong) X \times [0, 1)$.

Wong [Wo] showed that a Q -manifold X is $[0, 1)$ -stable if and only if X is properly contractible to ∞ ; that is, for any compactum K in X there is a proper map $j_K : X \rightarrow X \setminus K$ which is properly homotopic to id_X . Replacing a proper homotopy with a proper n -homotopy, we have the notion of *properly n -contractible to ∞* . Moreover we say that X is *properly locally (n)-contractible*

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at ∞ if for any compactum $K \subset X$ there is a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \rightarrow X \setminus L'$ which is properly (n -)homotopic to $\text{id}_{X \setminus L}$ in $X \setminus K$. In this paper we define μ_{∞}^{n+1} -manifolds as μ^{n+1} -manifolds which are properly n -contractible to ∞ and properly locally n -contractible at ∞ and show the following characterization theorem for μ_{∞}^{n+1} -manifolds.

Theorem I. *Let M be a μ^{n+1} -manifold. Then M is a μ_{∞}^{n+1} -manifold if and only if M is homeomorphic to its n -homotopy kernel $\text{Ker}(M)$.*

We will show that two n -homotopic proper maps into a μ_{∞}^{n+1} -manifold are properly n -homotopic (see Lemma 3.1). Thus we can remove the requirement of an n -homotopy between μ_{∞}^{n+1} -manifold to be proper, whence we obtain the following Z-set unknotting theorem for μ_{∞}^{n+1} -manifolds.

Theorem II. *Each homeomorphism between two Z-sets in a μ_{∞}^{n+1} -manifold M extends to an ambient homeomorphism of M onto itself if it is n -homotopic to id in M .*

From Theorem 2.2 in [Ch3], it follows that two μ_{∞}^{n+1} -manifolds of the same n -homotopy type are homeomorphic. Similarly to [C1, Theorem 5], we can clarify the relation between n -homotopy equivalences and homeomorphisms, that is:

Theorem III. *An n -homotopy equivalence between two μ_{∞}^{n+1} -manifolds is n -homotopic to a homeomorphism.*

Moreover, similarly to $[0, 1)$ -stable Q -manifolds [C1, Lemma 3.6], we can strengthen the open embedding theorem [Ch 2, Ch 3].

Theorem IV. *Each map from a μ_{∞}^{n+1} -manifold into a μ^{n+1} -manifold is n -homotopic to an open embedding.*

1. PRELIMINARIES

We say two (proper) maps $f, g : X \rightarrow Y$ are (properly) n -homotopic (notation: $f \simeq^n g$, $f \simeq_p^n g$, respectively) if, for any (proper) map $\alpha : Z \rightarrow X$ from a space Z with $\dim Z \leq n$ into X , the compositions $f\alpha$ and $g\alpha$ are (properly) homotopic in the usual sense. The notion of n -homotopy equivalence is defined in the obvious way.

Proposition 1.1 [Hu]. *Let $f : X \rightarrow Y$ be a map, where $\dim X \leq n$ and Y is LC^n . Then for any open cover $\mathcal{U}$ of Y , there are maps $\phi : X \rightarrow P$ and $\psi : P \rightarrow Y$ such that f and $\psi\phi$ are $\mathcal{U}$ -homotopic, where P is a locally finite polyhedron with $\dim P \leq n$. In particular, we can choose ψ as a proper map.*

Let us recall that a map $f : X \rightarrow Y$ is said to be n -invertible if for any space Z with $\dim Z \leq n$ and any map $\alpha : Z \rightarrow Y$ there exists a map $\beta : Z \rightarrow X$ such that $f\beta = \alpha$.

Proposition 1.2 [Ch2]. *Every μ^{n+1} -manifold admits a proper $(n+1)$ -invertible UV^n -surjection onto a Q -manifold.*

Proposition 1.3 [Ch3]. *Two μ^{n+1} -manifolds admitting proper UV^n -surjections onto the same LC^n -space are homeomorphic.*

The following theorem is due to Bestvina [Be], where it is stated in terms of μ -homotopy. However, as is known [Ch1], the notion of μ -homotopy coincides with one of n -homotopy for maps between locally compact LC^n -spaces of dimension at most $n + 1$.

Theorem 1.1 (Z-set unknotting theorem). *Let M be a μ^{n+1} -manifold and $f : A \rightarrow B$ be a homeomorphism between Z-sets in M . If $f \simeq_p^n \text{id}_A$ in M , then f extends to a homeomorphism $h : M \rightarrow M$.*

An n -homotopy kernel of a μ^{n+1} -manifold M is defined to be the complement $M \setminus f(M)$ of the image of an arbitrary Z-embedding $f : M \rightarrow M$ with $f \simeq_p^n \text{id}_M$. Using the Z-set unknotting theorem, two n -homotopy kernels are homeomorphic by an ambient homeomorphism of M onto itself. By $\text{Ker}(M)$, we denote a representative of n -homotopy kernels of M . The following proposition is actually proved in [Ch3].

Proposition 1.4. *For each μ^{n+1} -manifold M there exists a proper $(n + 1)$ -invertible UV^n -surjection $f : M \rightarrow M \times [0, 1]$ such that $f^{-1}(M \times [0, 1]) = \text{Ker}(M)$.*

Theorem 1.2 [Dr]. *There exists an $(n + 1)$ -invertible UV^n -surjection $f_n : \mu^{n+1} \rightarrow Q$ satisfying the following condition:*

(*) $f_n^{-1}(X)$ is a μ^{n+1} -manifold for any locally compact LC^n -space $X \subset Q$.

Theorem 1.3 [Ch4]. *For each locally finite polyhedron K , there exists a proper $(n + 1)$ -invertible UV^n -surjection $f_K : M_K \rightarrow K$ from a μ^{n+1} -manifold M_K onto K satisfying the following conditions:*

- (a) $f_K^{-1}(L)$ is a μ^{n+1} -manifold for any closed subpolyhedron L of K ;
- (b) $f_K^{-1}(Z)$ is a Z-set in $f_K^{-1}(L)$ for any Z-set Z in a closed subpolyhedron L of K .

Let $f : X \rightarrow Y$ be a proper map. We say that f induces an epimorphism of j^{th} homotopy groups of ends if for every compactum $C \subset Y$ there exists a compactum $K \subset Y$ such that for each point $x \in X \setminus f^{-1}(K)$ and every map $\alpha : (S^j, *) \rightarrow (Y \setminus K, f(x))$ there exist a map $\tilde{\alpha} : (S^j, *) \rightarrow (X \setminus f^{-1}(C), x)$ and a homotopy $f\tilde{\alpha} \simeq \alpha \text{ rel. } *$ in $Y \setminus C$. Also we say that f induces a monomorphism of j^{th} homotopy groups of ends if for every compactum $C \subset Y$ there exists a compactum $K \subset Y$ such that for every map $\tilde{\alpha} : S^j \rightarrow X \setminus f^{-1}(K)$ with $f\tilde{\alpha} \simeq *$ in $Y \setminus K$ it follows that $\tilde{\alpha} \simeq *$ in $X \setminus f^{-1}(C)$. It is said that f induces an isomorphism of j^{th} homotopy groups of ends if f induces both the epimorphism and monomorphism of j^{th} homotopy groups of ends.

Theorem 1.4 [Be]. *Let $f : M \rightarrow N$ be a proper map between μ^{n+1} -manifolds. If f induces an isomorphism of homotopy groups of $\dim \leq n$ and an isomorphism of homotopy groups of ends of $\dim \leq n$, then f is properly n -homotopic to a homeomorphism.*

2. CHARACTERIZATION OF μ_∞^{n+1} -MANIFOLDS

A space X is said to be properly (n) -contractible to ∞ if for any compactum K in X there exists a proper map $j_K : X \rightarrow X \setminus K$ which is properly

(n -)homotopic to id_X . If for any compactum $K \subset X$ there exists a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \rightarrow X \setminus L'$ which is properly (n -)homotopic to $\text{id}_{X \setminus L}$ in $X \setminus K$, then a space X is said to be *properly locally (n -)contractible at ∞* . It is easy to see that for any space X , $X \times [0, 1)$ is properly contractible to ∞ and properly locally contractible at ∞ .

Lemma 2.1. *Let X be properly n -contractible to ∞ and properly locally n -contractible at ∞ . Then for each compact cover $\{X_i\}_{i \in \omega}$ of X with $X_i \subset \text{int } X_{i+1}$, there exist a subcover $\{X_{i_k} | k \in \omega, 0 = i_0 < i_1 < i_2 < \dots\}$ and a collection of proper maps $\{f_k : X \rightarrow X \setminus X_{i_k}\}_{k \in \omega}$ such that $f_0 = \text{id}_X$ and $f_{k-1} \simeq_p^n f_k$ in $X \setminus X_{i_{k-2}}$ for $k \geq 1$, where $X_{i_{-1}} = \emptyset$.*

Proof. For technical reasons we assume that $X_0 = \emptyset$. Let $L_{-2} = L_{-1} = L_0 = \emptyset$. We shall inductively choose integers $0 = i_{-2} = i_{-1} = i_0 < i_1 < i_2 < \dots$ and construct compacta $L_{k-1} \subset X_{i_k} \subset L_k$ and proper maps $j_k : X \setminus L_{k-2} \rightarrow X \setminus X_{i_k}$, $k \in \omega$, satisfying the following conditions:

- (1) $j_0 = \text{id}_X$.
- (2) For each compactum $M \supset L_k$ there is a proper map $j_M : X \setminus L_k \rightarrow X \setminus M$ such that $j_M \simeq_p^n \text{id}_{X \setminus L_k}$ in $X \setminus X_{i_{k-2}}$.
- (3) $j_k \simeq_p^n \text{id}_{X \setminus L_{k-2}}$ in $X \setminus X_{i_{k-2}}$.

Let $i_1 = 1$. Since X is properly n -contractible to ∞ and properly locally n -contractible at ∞ , there exist a proper map $j_1 : X \rightarrow X \setminus X_{i_1}$ with $j_1 \simeq_p^n \text{id}$ and a compactum $L_1 \supset X_1$ satisfying (2). Since $X = \bigcup_{i \in \omega} X_i$ and $X_i \subset \text{int } X_{i+1}$, there exists $i_2 > i_1$ such that $X_{i_2} \supset L_1$. As in the above arguments there exist a proper map $j_2 : X \rightarrow X \setminus X_{i_2}$ with $j_2 \simeq_p^n \text{id}_X$ and a compactum $L_2 \supset X_{i_2}$ satisfying (2).

Assume that, for $k \geq 2$, $i_0 < i_1 < \dots < i_k$, L_k , and $j_k : X \setminus L_{k-2} \rightarrow X \setminus X_{i_k}$ have been constructed. Choose $i_{k+1} > i_k$ so that $X_{i_{k+1}} \supset L_k$. Since $X_{i_{k+1}} \supset L_{k-1}$, by the property (2) of L_{k-1} , there exists a proper map $j_{X_{i_{k+1}}} : X \setminus L_{k-1} \rightarrow X \setminus X_{i_{k+1}}$ such that $j_{X_{i_{k+1}}} \simeq_p^n \text{id}_{X \setminus L_{k-1}}$ in $X \setminus X_{i_{k-1}}$. Then put $j_{k+1} = j_{X_{i_{k+1}}}$. Since X is properly locally n -contractible at ∞ , there exists a compactum $L_{k+1} \supset X_{i_{k+1}}$ satisfying (2).

Now define $f_k = j_k \cdots j_0 : X \rightarrow X \setminus X_{i_k}$ for $k \in \omega$ and observe that the collections of compacta $\{X_{i_k}\}_{k \in \omega}$ and maps $\{f_k\}_{k \in \omega}$ are as desired. $\square$

As is stated in the introduction, a μ_∞^{n+1} -manifold is a μ^{n+1} -manifold which is properly n -contractible to ∞ and properly locally n -contractible at ∞ . Theorem I is contained in the following.

Theorem 2.1 (Characterization). *For a μ^{n+1} -manifold M the following conditions are equivalent:*

- (1) M is a μ_∞^{n+1} -manifold.
- (2) $M \cong \text{Ker}(M)$.
- (3) There is a proper $(n+1)$ -invertible UV^n -surjection $f : M \rightarrow X$ onto some $[0, 1)$ -stable Q -manifold X .
- (4) There is a proper $(n+1)$ -invertible UV^n -surjection $g : M \rightarrow Y$ onto a space Y which is properly n -contractible to ∞ and properly locally n -contractible at ∞ .

Proof. We shall prove that (1) $\Rightarrow$ (2). First we shall choose a compact cover $\{M_i\}_{i \in \omega}$ of M with $M_i \subset \text{int } M_{i+1}$, $i \in \omega$, such that the topological frontier $\text{Fr } M_i$ is a Z -set in $M \setminus \text{int } M_i$. To this end, fix a proper UV^n -surjection $g : M \rightarrow X$ onto a Q -manifold X . Then choose a compact cover $\{X_i\}_{i \in \omega}$ of X consisting of Q -manifold with $X_i \subset \text{int } X_{i+1}$ such that $\text{Fr } X_i$ is a Z -set in both X_i and $X \setminus \text{int } X_i$, $i \in \omega$ (see [C2, CS]). For each $i \in \omega$, by the relative triangulation theorem for Q -manifolds [C3], we may assume that $X = P \times Q$, $X_i = P_1^i \times Q$, and $X \setminus \text{int } X_i = P_2^i \times Q$ for a locally finite polyhedron P and closed subpolyhedra P_1^i , $P_2^i \subset P$. Note that $P_1^i \cap P_2^i$ is a Z -set in P_2^i . Let $f_P : M_P \rightarrow P$ be a proper UV^n -surjection from a μ^{n+1} -manifold M_P onto P satisfying condition (b) in Theorem 1.3. Since the composition $\pi_P g : M \rightarrow P$ is proper UV^n (where $\pi_P : P \times Q \rightarrow P$ is the canonical projection), there is a homeomorphism $k : M_P \rightarrow M$ by Proposition 1.3. Then by property (b) of f_P , $f_P^{-1}(P_1^i \cap P_2^i)$ is a Z -set in $f_P^{-1}(P_2^i)$ and so is the topological frontier of $f_P^{-1}(P_1^i)$. Now let $M_i = k f_P^{-1}(P_1^i)$, $i \in \omega$. Then the compact cover $\{M_i\}_{i \in \omega}$ of M is the required one.

By Lemma 2.1, there is a collection of maps $\{f_i : M \rightarrow M \setminus M_i\}_{i \in \omega}$ such that $f_0 = \text{id}_M$, $f_i \simeq_p^n f_{i+1}$ in $M \setminus M_{i-1}$ for $i \in \omega$. Using the Z -embedding approximation theorem for μ^{n+1} -manifolds [Be, 2.3.8], we can choose f_i as a Z -embedding for each $i \in \omega$. Put $K_i = M \setminus f_i(M)$ for $i \geq 1$. Then since $f_i \simeq_p^n \text{id}_M$, by the definition of n -homotopy kernels, we have $K_i \cong \text{Ker}(M)$. By Theorem 1.1, since $f_i \simeq_p^n f_{i+1}$ in $M \setminus M_{i-1}$ and $\text{Fr } M_{i-1}$ is a Z -set in $M \setminus \text{int } M_{i-1}$, there exists a homeomorphism $h_i : M \rightarrow M$ such that $h_i f_i = f_{i+1}$ and $h_i|_{M_{i-1}} = \text{id}$. Note that $h_i(K_i) = K_{i+1}$. Now we define $h : K_1 \rightarrow M$ by $h = \lim_{i \rightarrow \infty} h_i \cdots h_1$. Then $h|_{h^{-1}(\text{int } M_i)} = h_{i+2} \cdots h_1|_{h^{-1}(\text{int } M_i)}$. In fact, suppose that $h(x) \neq h_{i+2} \cdots h_1(x)$ for some $x \in h^{-1}(\text{int } M_i)$. Then there is an open subset U of $\text{int } M_i$ such that $h(x) \in U \subset \bar{U}$ and $h_{i+2} \cdots h_1(x) \notin \bar{U}$. Since $h_j|_{\text{int } M_i} = \text{id}$ for $j \geq i+2$, $h_j \cdots h_1(x) = h_{i+2} \cdots h_1(x) \notin \bar{U}$ for all $j \geq i+2$. This contradicts the definition of h .

One can easily see that h is injective. Moreover, since $M = \bigcup_{i \in \omega} M_i$ and $h_i \cdots h_1(K_1) = K_i \supset M_i$, it follows that h is surjective. To finish the proof, it only remains to note that h is open. Thus h is a homeomorphism.

To prove (2) $\Rightarrow$ (3), assume $M \cong \text{Ker}(M)$. Then, by Proposition 1.4, there is a proper $(n+1)$ -invertible UV^n -surjection $g : M \rightarrow M \times [0, 1)$. Let $h : M \rightarrow Y$ be a proper UV^n -surjection onto a Q -manifold Y (Proposition 1.2). Since $Y \times [0, 1)$ is a $[0, 1)$ -stable Q -manifold, the composition $(h \times \text{id}_{[0, 1)})g : M \rightarrow Y \times [0, 1)$ is the required one.

(3) $\Rightarrow$ (4) is trivial.

Finally we shall show that (4) $\Rightarrow$ (1). Let $h : M \rightarrow X$ be a proper $(n+1)$ -invertible UV^n -surjection onto a space X properly n -contractible to ∞ and properly locally n -contractible at ∞ . Let K be a compactum in M . Then there exists a compactum L' in X with $h(K) \subset L'$ such that for each compactum F' with $L' \subset F'$ there exist proper maps $i'_{h(K)} : X \rightarrow X \setminus h(K)$ and $j'_{F'} : X \setminus L' \rightarrow X \setminus F'$ such that $i'_{h(K)} \simeq_p^n \text{id}_X$ in X and $j'_{F'} \simeq_p^n \text{id}_{X \setminus L'}$ in $X \setminus h(K)$. Let $L = h^{-1}(L')$ and F be a compactum containing L . Since h is proper $(n+1)$ -invertible, there exist proper maps $i_K : M \rightarrow M \setminus K$ and $j_F : M \setminus L \rightarrow M \setminus F$ such that $h i_K = i'_{h(K)} h$ and $h j_F = j'_{h(F)} h$.

Consider a proper map $\alpha : Z \rightarrow M \setminus L \ (\subset M \setminus h^{-1}h(K))$, where $\dim Z \leq n$. We shall now show that $j_F \alpha$ is properly homotopic to α in $M \setminus K$. From Proposition 1.1, we may assume without loss of generality that Z is a locally finite polyhedron. Let $H : (X \setminus L') \times [0, 1] \rightarrow X \setminus h(K)$ be a proper homotopy from $\text{id}_{X \setminus L'}$ to $j'_{h(F)}$. Then $H(h\alpha \times \text{id}) : Z \times [0, 1] \rightarrow X \setminus h(K)$ is a proper homotopy from $h\alpha$ to $j'_{h(F)}h\alpha = hj_F\alpha$. Since $h|_{M \setminus h^{-1}h(K)} : M \setminus h^{-1}h(K) \rightarrow X \setminus h(K)$ is proper UV^n , by [La, §3, Lemma A], there exists a proper homotopy $F : Z \times [0, 1] \rightarrow M \setminus h^{-1}h(K)$ from α to $j_F\alpha$. Thus $j_F \simeq_p^n \text{id}_{M \setminus L}$ in $M \setminus K$. Similarly, we can conclude $i_F \simeq_p^n \text{id}_M$. $\square$

3. PROOFS OF THEOREMS II, III, AND IV

Lemma 3.1. *Let $f : X \rightarrow Y$ be a map from a locally compact space X into a LC^n -space Y admitting a proper $(n+1)$ -invertible UV^n -surjection onto a space $Y \times [0, 1]$. Then f is n -homotopic to a proper map whenever $\dim X \leq n+1$. Moreover, if f is a proper map n -homotopic to a proper map $g : X \rightarrow Y$, then $f \simeq_p^n g$.*

Proof. Fix a proper map $p : X \rightarrow [0, 1]$, and let $h : Y \rightarrow Y \times [0, 1]$ be a proper $(n+1)$ -invertible UV^n -surjection. Let $q : X \rightarrow Y \times [0, 1]$ be the map defined by $q(x) = (h_1 f(x), p(x))$, where $h(x) = (h_1(x), h_2(x))$, $x \in X$. Then q is proper and homotopic to hf . By the $(n+1)$ -invertibility of h , there is a map $f' : X \rightarrow Y$ such that $hf' = q$. Note that f' is proper and $hf' \simeq hf$. Thus by the lifting property of h [La, §3, Lemma A], we conclude that $f \simeq^n f'$.

Next suppose that f is a proper map n -homotopic to a proper map $g : X \rightarrow Y$. Let $\alpha : Z \rightarrow X$ be a proper map, where $\dim Z \leq n$. We shall show that $f\alpha \simeq_p g\alpha$. By Proposition 1.1, we may assume without loss of generality that Z is a locally finite polyhedron. Let $\{Y_i\}_{i \in \omega}$ be a compact cover of Y with $Y_0 = \emptyset$ and $Y_i \subset \text{int } Y_{i+1}$, $i \in \omega$. Then for each $i \geq 1$, let Z_i be a compact subpolyhedron of Z such that $(hf\alpha)^{-1}(W_i) \cup (hg\alpha)^{-1}(W_i) \subset Z_i \subset \text{int } Z_{i+1}$, where $Z_0 = \emptyset$ and $W_i = Y_i \times [0, 1 - 2^{-i}]$. Since $f \simeq^n g$, we can fix a homotopy $G_0 : Z \times [0, 1] \rightarrow Y$ from $f\alpha$ to $g\alpha$. For $k \geq 1$, we shall inductively construct a homotopy $G_k : (Z \setminus \text{int } Z_{2k-2}) \times [0, 1] \rightarrow Y \setminus h^{-1}(W_{2k-5})$ from the restriction $f\alpha|$ of $f\alpha$ to the one $g\alpha|$ of $g\alpha$ satisfying the following conditions:

- (1)_k $G_k((Z \setminus \text{int } Z_{2k}) \times [0, 1]) \subset Y \setminus h^{-1}(W_{2k-2})$;
- (2)_k $G_k = G_{k-1}$ on $\text{Fr } Z_{2k-2} \times [0, 1]$.

Let $F_i : [0, 1] \rightarrow [1 - 2^{-i}, 1]$ be the map defined by $F_i(t) = 1 + (t - 1)2^{-i}$ for each $i \geq 1$. Suppose that a homotopy $G_k : (Z \setminus \text{int } Z_{2k-2}) \times [0, 1] \rightarrow Y \setminus h^{-1}(W_{2k-5})$ has been constructed for $k \in \omega$. Then let $A_{k+1} = (Z \setminus \text{int } Z_{2k}) \times \{0, 1\} \cup \text{Fr } Z_{2k} \times [0, 1]$ and $B_{k+1} = (hG_k)^{-1}(W_{2k+1}) \cap (Z \setminus \text{int } Z_{2k+2}) \times [0, 1]$. Since A_{k+1} and B_{k+1} are disjoint closed, we can choose $\beta : (Z \setminus \text{int } Z_{2k}) \times [0, 1] \rightarrow [0, 1]$ such that $\beta(A_{k+1}) = 0$ and $\beta(B_{k+1}) = 1$. Define $G'_{k+1} : (Z \setminus \text{int } Z_{2k}) \times [0, 1] \rightarrow Y \times [0, 1] \setminus W_{2k-2}$ by

$$G'_{k+1}(w) = (s_k(w), (1 - \beta(w))t_k(w) + \beta(w)F_{2k+2}t_k(w)),$$

where $hG_k(w) = (s_k(w), t_k(w))$, $w \in (Z \setminus \text{int } Z_{2k}) \times [0, 1]$. By the lifting property [La], there is a homotopy $G_{k+1} : (Z \setminus \text{int } Z_k) \times [0, 1] \rightarrow Y \setminus W_{2k-3}$ from $f\alpha|$ to $g\alpha|$ with $hG_{k+1} = G'_{k+1}$ and $G_{k+1} = G_k$ on A_{k+1} (i.e., satisfying (2)_{k+1}) such that G_{k+1} satisfies (1)_{k+1}.

We define $H : Z \times [0, 1] \rightarrow Y$ by $H = G_k$ on each $(Z_{2k} \setminus \text{int } Z_{2k-2}) \times [0, 1]$. Then H is a well-defined homotopy from f_α to g_α . Note that since h is proper, $\{h^{-1}(W_i)\}_{i \in \omega}$ is a compact cover of Y with $h^{-1}(W_i) \subset \text{int } h^{-1}(W_{i+1})$. Thus it follows from our construction that H is proper. The proof is finished. $\square$

Proof of Theorem II. The theorem directly follows from Theorem 1.1 and Lemma 3.1. $\square$

Lemma 3.2. *If $f : M \rightarrow N$ is a proper n -homotopy equivalence between μ_∞^{n+1} -manifolds, then f induces an isomorphism of homotopy groups of ends of $\dim \leq n$.*

Proof. By Theorem 2.1, we can fix proper $(n+1)$ -invertible UV^n -surjections $g : M \rightarrow X \times [0, 1)$ and $h : N \rightarrow Y \times [0, 1)$, where X and Y are some Q -manifolds. Let C be a compactum in N . Then there is a compactum $C'' \subset Y$ such that $C'' \times [0, t'] \supset h(C)$ for some $t' \in (0, 1)$. Since h is proper, $C' = h^{-1}(C'' \times [0, t'])$ is a compactum with $C' \supset C$. Note that, since f is proper, $g(f^{-1}(C'))$ is a compactum in $X \times [0, 1)$. Thus there exists $t_1 \in (0, 1)$ such that $L = \pi_X(g(f^{-1}(C'))) \times [0, t_1] \supset g(f^{-1}(C'))$, where $\pi_X : X \times [0, 1) \rightarrow X$ is the canonical projection. Similarly, since g is proper, there exists $t_2 \in (0, 1)$ such that $K' = \pi_Y(hf(g^{-1}(L))) \times [0, t_2] \supset hf(g^{-1}(L))$, where $\pi_Y : Y \times [0, 1) \rightarrow Y$ is the canonical projection. Put $K = h^{-1}(K')$, and let $x_0 \in M \setminus f^{-1}(K)$, $j \leq n$, and $\alpha : (S^j, *) \rightarrow (N \setminus K, f(x_0))$. Since f is an n -homotopy equivalence, there exists $\alpha_1 : (S^j, *) \rightarrow (M, x_0)$ such that $f\alpha_1 \simeq \alpha \text{ rel. } *$. Since $\alpha_1^{-1}(x_0)$ and $\alpha_1^{-1}(g^{-1}(L))$ are disjoint closed sets in S^j , we can choose a map $\beta : S^j \rightarrow [0, 1]$ such that $\beta(\alpha_1^{-1}(x_0)) = 0$ and $\beta(\alpha_1^{-1}(g^{-1}(L))) = 1$. Say $g\alpha_1(x) = (\pi_X g\alpha_1(x), t(x)) \in X \times [0, 1)$, $x \in S^j$. Define $\alpha_2 : (S^j, *) \rightarrow (X \times [0, 1), g(x_0))$ by

$$\alpha_2(x) = (\pi_X g\alpha_1(x), (1 - t_1) \cdot t(x) + t_1)\beta(x) + (1 - \beta(x) \cdot t(x)), \quad x \in S^j.$$

Clearly $\alpha_2 \simeq g\alpha_1 \text{ rel. } *$ and $\alpha_2(S^j) \cap L = \emptyset$. Using the lifting property [La] of the proper UV^n -surjection g , there exists $\tilde{\alpha} : (S^j, *) \rightarrow (M, x_0)$ such that $\text{img } \tilde{\alpha} \cap L = \emptyset$ and $\tilde{\alpha} \simeq \alpha_1 \text{ rel. } *$. Hence we have $f\tilde{\alpha} \simeq \alpha \text{ rel. } *$ and $f\tilde{\alpha}(S^j) \cap C' = \emptyset$. By the same technique we performed above, we can choose a homotopy so that $f\tilde{\alpha} \simeq \alpha \text{ rel. } *$ in $N \setminus C$.

Next let $\gamma : S^j \rightarrow M \setminus f^{-1}(K)$ be a map such that $f\gamma \simeq *$ in $N \setminus K$. Since f is an n -homotopy equivalence, $g\gamma \simeq *$ in $X \times [0, 1)$. By sliding the $[0, 1)$ -factor of the homotopy upward as the above, we have $g\gamma \simeq *$ in $X \times [0, 1) \setminus L$. By the lifting property of g [La], it follows that $\gamma \simeq *$ in $X \setminus f^{-1}(C)$. Thus we conclude that f induces an isomorphism of homotopy groups of ends of $\dim \leq n$. $\square$

Proof of Theorem III. Let $f : M \rightarrow N$ be an n -homotopy equivalence between μ_∞^{n+1} -manifolds. Then by Lemma 3.1 there is a proper map $h : M \rightarrow N$ such that $f \simeq^n h$; consequently, h is a proper n -homotopy equivalence. By Lemma 3.2 and Theorem 1.4, h is properly n -homotopic to a homeomorphism. Thus f is n -homotopic to a homeomorphism. $\square$

Proof of Theorem IV. Let $f : M \rightarrow N$ be a map from a μ_∞^{n+1} -manifold to a μ_∞^{n+1} -manifold. By replacing N with $\text{Ker}(N)$, we may assume that N is also a μ_∞^{n+1} -manifold. By the triangulation theorem for μ_∞^{n+1} -manifold [Dr], we can fix proper $(n+1)$ -invertible UV^n -surjections $g : M \rightarrow K$ and $h : N \rightarrow L$, where K and L are locally finite polyhedra of dimension at most $n+1$. Then by the $(n+1)$ -invertibility, g has a section $p : K \rightarrow M$ (i.e., $gp = \text{id}_K$). Since N is a μ_∞^{n+1} -manifold, by Lemma 3.1, f is n -homotopic to a proper map $f' : M \rightarrow N$. Then $\varphi = hf'p : K \rightarrow L$ is a proper map. Let $M(\varphi)$ be the mapping cylinder of φ , that is, a space obtained from the disjoint union $K \times [0, 1] \oplus L$ by identifying $(x, 1)$ with $\varphi(x)$, $x \in K$. Define $c_\varphi : M(\varphi) \rightarrow L$ by $c_\varphi(x, t) = \varphi(x)$, $x \in K$. Let $f_n : \mu_\infty^{n+1} \rightarrow Q$ be a proper $(n+1)$ -invertible UV^n -surjection satisfying the condition (*) in Theorem 1.2. Embed $M(\varphi)$ into Q , whence $f_n^{-1}(M(\varphi))$ is a μ_∞^{n+1} -manifold. We denote the restriction of f_n to $f_n^{-1}(M(\varphi))$ by $f_n|$. Observe that $f_n^{-1}(K \times \{0\}) \cong M$ and $f_n^{-1}(L) \cong N$ by Proposition 1.3. We identify $f_n^{-1}(K \times \{0\})$, $f_n^{-1}(L)$ with M , N respectively. Abusing notation, by $g : M \rightarrow K \times \{0\}$, $h : N \rightarrow L$ we denote the restrictions of f_n to M , N respectively. Using the $(n+1)$ -invertibility of h , we can fix a section $q : L \rightarrow N$ of h . Note that since $c_\varphi f_n| : f_n^{-1}(M(\varphi)) \rightarrow L$ and $h : N \rightarrow L$ are proper UV^n -surjections, $f_n^{-1}(M(\varphi)) \cong N$ by Proposition 1.3. Observe that the map $qc_\varphi f_n|$ is an n -homotopy equivalence between μ_∞^{n+1} -manifolds $f_n^{-1}(M(\varphi))$ and N . Then by Theorem III, there is a homeomorphism $s : f_n^{-1}(M(\varphi)) \rightarrow N$ such that $s \simeq^n qc_\varphi f_n|$. Note that $M' = f_n^{-1}(K \times [0, 1])$ is open in $f_n^{-1}(M(\varphi))$ and is a μ_∞^{n+1} -manifold by Theorem 2.1. Since the inclusion $i : M = f_n^{-1}(K \times \{0\}) \hookrightarrow M'$ is an n -homotopy equivalence, by Theorem III, we can choose a homeomorphism $r : M \rightarrow M'$ with $r \simeq^n i$. Then the map $sr : M \rightarrow N$ is an open embedding which is n -homotopic to $qc_\varphi(f_n|i) = q\varphi g = qhf'pg$. Since $pg \simeq_p^n \text{id}_M$ and $qh \simeq_p^n \text{id}_N$, we have $qhf'pg \simeq_p^n f' \simeq^n f$. The proof is finished. $\square$

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