BUILDINGS AS INNER IDEAL GEOMETRIES

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ABSTRACT. We consider a Chevalley-type group $G(V)$ associated to an integrable representation of a Kac-Moody algebra and show that the associated Kac-Moody group $G(A)$ is a universal cover for $G(V)$. This observation strengthens a result of Kac-Peterson on representations of $G(A)$. It also implies that the building associated to an affine Lie algebra can be realized as an inner ideal geometry.

We refer the reader to [4] for details concerning Kac-Moody algebras and their representations.

Let S be a finite index set, and let $\mathfrak{g}$ be the Kac-Moody algebra associated to a generalized Cartan matrix, $A = (a_{ij})_{i,j \in S}$. We assume throughout that A is indecomposable.

Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$. Its linear dual is $\mathfrak{h}^*$. Use Π to designate the set of simple roots, Δ the set of roots, Δ^{re} the real roots, and Δ_+^{re} the positive real roots of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be its root space decomposition. Fix a set of Chevalley generators for $\mathfrak{g}$, $\{e_i, f_i : i \in S\}$ where $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ for $\alpha_i \in \Pi$. When $\mathfrak{g}$ is affine, we use $(\cdot, \cdot)$ to designate a standard invariant form.

Kac-Moody groups and buildings. We consider two types of groups associated to $\mathfrak{g}$ and its representations. The first is defined in terms of A without reference to $\mathfrak{g}$ at all. This is the Kac-Moody group which we designate $G(A)$. It is constructed directly as follows. (This definition is from [8]. See [5] for an axiomatic treatment.)

Let G^* be the free product of the additive groups $\mathfrak{g}_\alpha$ for $\alpha \in \Delta^{re}$. Denote the canonical injections by $\iota_\alpha : \mathfrak{g}_\alpha \rightarrow G^*$. If $(V, d\pi)$ is an integrable representation of $\mathfrak{g}$, then we have a representation $\pi^* : G^* \rightarrow \text{Aut}_{\mathbb{C}}(V)$ defined by $\pi^*(\iota_\alpha(x_\alpha)) = \exp d\pi(x_\alpha)$. (Since x_α is locally nilpotent, $\exp d\pi(x_\alpha)$ is a well-defined automorphism of V .) Define $G(A) = G^*/N^*$ for $N^* = \bigcap \ker \pi^*$ where the intersection is taken over all integrable representations of $\mathfrak{g}$.

As noted in [5], there is an injective homomorphism $\varphi_i : SL(2, \mathbb{C}) \rightarrow G(A)$ for each $i \in S$. The set of all $G_i = \varphi_i(SL(2, \mathbb{C}))$ generates $G(A)$.

Induce a topology on $G(A)$ by decreeing that the maps φ_i be continuous with respect to the usual topology on $SL(2, \mathbb{C})$. The Kac-Moody group is then

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a connected, simply connected topological group which satisfies the Hausdorff separation axiom (cf. [6]).

It is well known [5–8] that $G(A)$ has a BN -pair structure (also called a Tits system; cf. [1, Chapter IV, §2.1]). We take our BN -pair for $G(A)$ to be as given in [8]. Here we write $B = B(A)$ and $N = N(A)$. Note that when $b \in B(A)$ and V is an integrable highest weight representation of $\mathfrak{g}$ with weight space V_λ , $b \cdot V_\lambda \subseteq \sum V_\rho$ where $\rho \geq \lambda$. In other words, $\rho - \lambda = \sum_{\alpha_i \in \Pi} k_i \alpha_i$ where $k_i \geq 0$ for each i .

Fix an integrable representation of $\mathfrak{g}$, $d\pi : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(V)$. Let $\pi : G(A) \rightarrow \text{Aut}_{\mathbb{C}}(V)$ be given by $\pi(g) = \pi^*(g^*)$ for $g = g^*N^* \in G(A)$, $g^* \in G^*$. The second group we consider is $G(V)$, the homomorphic image of $G(A)$ under π . With the quotient topology, $G(V)$ is a connected topological group.

Define $\varphi'_i : SL(2, \mathbb{C}) \rightarrow G(V)$ by $\varphi'_i = \pi \circ \varphi_i$ for each $i \in S$. Note that φ'_i is continuous for all $i \in S$.

If $G'_i = \varphi'_i(SL(2, \mathbb{C}))$, then for each $i \in S$, G'_i is either trivial, isomorphic to $SL(2, \mathbb{C})$, or isomorphic to $PSL(2, \mathbb{C})$.

Theorem 1. *Let $(V, d\pi)$ be an integrable representation for a Kac-Moody algebra $\mathfrak{g}$. If G'_i is nontrivial for some $i \in S$, then $G(A)$ is a universal covering group and a central extension of $G(V)$.*

Proof. We have our continuous surjective homomorphism, $\pi : G(A) \rightarrow G(V)$. Since $G(A)$ is simply connected, we can prove that $G(A)$ is a universal cover by showing that π is a local homeomorphism. Indeed, it suffices to show that π is injective on some neighborhood of the identity.

Pick $i \in S$ so that G'_i is nontrivial. Since $G'_i \cong SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C})$, $\varphi'_i = \pi \circ \varphi_i$ is a local homeomorphism. In particular, π is injective on some neighborhood of 1 in $\varphi_i(SL(2, \mathbb{C})) \subseteq G(A)$. This shows that $G(A)$ is a universal cover of $G(V)$. By [10, §7, Example (c)], our proof is now complete. $\square$

Let $C(G)$ denote the center of a group G .

The next corollary strengthens Corollaries 4.4(a) and 4.5 of [5].

Corollary. *If $(V, d\pi)$ is an integrable representation of a Kac-Moody algebra $\mathfrak{g}$, then $\ker \pi = G(A)$ or $\ker \pi \subseteq C(G(A))$.*

Proof. If G'_i is trivial for all $i \in S$, then by definition of φ'_i and the fact that $G(A)$ is generated by G_{iS} , π is trivial. In this case, $\ker \pi = G(A)$.

If G'_i is nontrivial for some $i \in S$, then Theorem 1 applies to give us $\ker \pi \subseteq C(G(A))$. $\square$

Assume now that $(V, d\pi)$ is an integrable representation of $\mathfrak{g}$ with the property that $d\pi(x)$ is nonzero for some $x \notin \mathfrak{h}$. Let $B(V) = \pi(B(A))$ and $N(V) = \pi(N(A))$. Since $\ker \pi \subseteq H(A)$, the following is immediate by Remark (2) in [1, Chapter IV, §2.1].

Theorem 2. *$(G(V), B(V), N(V))$ is a group with BN -pair.*

A building is a geometric configuration we associate to a group with BN -pair as follows. (Consult [2, 9] for the details.)

If (G, B, N) is a group with BN -pair, then $W = B/N$ is a Coxeter group, generated by reflections r_i for i in some index set I . Any subgroup of G which contains B is called special. The special subgroups of G , it turns out, are

precisely those of the form $BW'B$ where W' is a subgroup of W generated by reflections r_i for $i \in I' \subseteq I$. The G -conjugates of special subgroups are called the parabolic subgroups of G . The parabolics, partially ordered by opposite inclusion, comprise the building we associate to (G, B, N) .

The building we get from $(G(A), B(A), N(A))$ is the one we associate to $\mathfrak{g}$.

Fix $(V, d\pi)$, an integrable representation of $\mathfrak{g}$ such that $d\pi(x) \neq 0$ for some $x \notin \mathfrak{h}$. Since $G(A)$ is a central extension of $G(V)$ and $C(G(A)) \subseteq H(A) \subseteq B(A)$ (cf. [5]), the buildings associated to $(G(A), B(A), N(A))$ and $(G(V), B(V), N(V))$ are actually isomorphic. Note also that the Coxeter group associated to $(G(V), B(V), N(V))$ is isomorphic to W , the Weyl group associated to $\mathfrak{g}$. In this case, the reflections which generate $B(V)/N(V)$ can be identified with the reflections in $\mathfrak{h}^*$ determined by $\alpha_i \in \Pi$. We identify W with $B(V)/N(V)$ in this setting so that if V is a highest weight module with weight space V_λ , $w \cdot V_\lambda = V_{w(\lambda)}$ for all $w \in W$.

Buildings as inner ideal geometries. Now we assume that $\mathfrak{g}$ is affine.

Let $(V, d\pi)$ be a faithful irreducible highest weight representation of $\mathfrak{g}$. We define an inner ideal in V as in [3]. By the inner ideal geometry associated to $(V, d\pi)$ we mean the inner ideals in V partially ordered by opposite inclusion.

Suppose that V has highest weight Λ . If $\lambda = \Lambda - \sum_{\alpha_i \in \Pi} k_i \alpha_i$ is a weight of V , the support of λ , $\text{supp } \lambda$, is the set of simple roots α_i with $k_i \neq 0$. When $\Pi' \subseteq \Pi$, we designate by $R(\Pi')$ the set of weights λ with $\text{supp } \lambda \cap \Pi' = \emptyset$.

The notion of a set of roots which is reduced mod Λ is discussed in [3]. There we show that when $\Pi' \subseteq \Pi$ is reduced mod Λ , $\mathcal{J}(\Pi') = \sum_{\lambda \in R(\Pi')} V_\lambda$ is an inner ideal. Every inner ideal in V , moreover, is a $G(V)$ -translate of $\mathcal{J}(\Pi')$ for some reduced set of roots Π' . When $\mathcal{J}$ is an inner ideal in V and we write $\mathcal{J} = g \cdot \mathcal{J}(\Pi')$, it should be understood that g is in $G(V)$ and $\Pi' \subseteq \Pi$ is reduced mod Λ .

Let $\Pi_\Lambda = \{\alpha_i \in \Pi : (\Lambda, \alpha_i) \neq 0\}$. Assume that $\Pi = \Pi_\Lambda$ so that every subset of Π is reduced mod Λ . Our next result says that the parabolic subgroups of $G(V)$ are the stabilizers of the inner ideals in V .

We designate the special subgroup $B(V) < r_i : i \notin S' > B(V)$ by $P_{S'}$.

Theorem 3. *Let $\mathcal{J} = g \cdot \mathcal{J}(\Pi')$ be an inner ideal in V with $\Pi' = \{\alpha_i \in \Pi : i \in S'\}$. The stabilizer of $\mathcal{J}$ in $G(V)$ is $gP_{S'}g^{-1}$.*

Proof. Take $\lambda \in R(\Pi')$ so that $V_\lambda \subseteq \mathcal{J}(\Pi')$. For $b \in B(V)$, $b \cdot V_\lambda \subseteq \sum_{\rho} V_\rho$ where $\text{supp } \rho \subseteq \text{supp } \lambda$. It follows that all such weights ρ belong to $R(\Pi')$ thus that $B(V)$ stabilizes $\mathcal{J}(\Pi')$. Since any subgroup of $G(B)$ which contains $B(V)$ is special, it follows that the stabilizer of $\mathcal{J}(\Pi')$ in $G(V)$ is special.

Let j belong to the complement of S' in S . Observe that for any $\lambda \in R(\Pi')$, $\text{supp } r_j(\lambda) \cap S' = \emptyset$. It follows that r_j stabilizes $\mathcal{J}(\Pi')$. On the other hand, if j belongs to S' , then $\text{supp } r_j(\Lambda) \cap S' = \{\alpha_j\}$ since $(\Lambda, \alpha_j) \neq 0$. As $\Lambda \in R(\Pi')$ for all $\Pi' \subseteq \Pi$, it follows that r_j stabilizes $\mathcal{J}(\Pi')$ if and only if $j \notin S'$. This gives us that $P_{S'}$ is the stabilizer of $\mathcal{J}(\Pi')$.

Let $\mathcal{J} = g \cdot \mathcal{J}(\Pi')$ be an arbitrary inner ideal in V . It is clear that the elements in $gP_{S'}g^{-1}$ stabilize $\mathcal{J}$. Suppose that $g' \in G(V)$ stabilizes $\mathcal{J}$. We then have $g^{-1}g'g$ stabilizing $\mathcal{J}(\Pi')$ so that $g^{-1}g'g \in P_{S'}$. This proves the theorem. $\square$

Our last result comes as a straightforward corollary to Theorem 3.

Corollary. *The building associated to an affine Lie algebra can be realized as an inner ideal geometry.*

Proof. Take V to be an irreducible, integrable highest weight representation of $\mathfrak{g}$ with highest weight Λ satisfying $\Pi_\Lambda = \Pi$. Let f map the inner ideals in V to the parabolic subgroups of $G(V)$ by sending each inner ideal to its stabilizer. This is a well-defined function: if $g_1 \cdot \mathcal{I}(\Pi_1) = g_2 \cdot \mathcal{I}(\Pi_2)$, then Lemma 3.2 in [3] implies that $\Pi_1 = \Pi_2$ so that $P_{S_1} = P_{S_2}$. From here we see that $g_1^{-1}g_2$ belongs to P_{S_1} so that $g_1 P_{S_1} g_1^{-1} = g_2 P_{S_2} g_2^{-1}$.

We can also show that f is bijective. Surjectivity is clear. By Theorem 2 in [2, Chapter V, §2B], distinct special subgroups cannot be conjugate. This implies that f is injective.

Since the partial ordering on inner ideals and the partial ordering on parabolic subgroups are both defined as opposite inclusion of sets, f is an isomorphism from the geometry of inner ideals in V to the building determined by $\mathfrak{g}$. The building associated to $\mathfrak{g}$ can thus be realized as the inner ideal geometry associated to $(V, d\pi)$. $\square$

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