A TOPOLOGICAL EQUIVALENCE OF THE SINGULAR CARDINALS HYPOTHESIS

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ABSTRACT. We show, in particular, that the Singular Cardinals Hypothesis is equivalent to the following statement: for each metric space X of cardinality greater than the continuum and of weight of uncountable cofinality, the cardinality of X equals the weight of X .

The *Singular Cardinals Hypothesis*, SCH, states that if κ is a singular cardinal and $2^{\text{cf}(\kappa)} < \kappa$, then $\kappa^{\text{cf}(\kappa)} = \kappa^+$ (cf. [Je]). The aim of this note is to give a purely topological equivalence of SCH.

Given a topological space X , let $w(X)$, $d(X)$, $s(X)$, and $t(X)$ denote, respectively, the weight of X , the density of X , the spread of X , and the tightness of X . For definitions of these and other concepts see [Ju].

Let P denote the class of all regular T_1 topological spaces X which satisfy the following two conditions:

- (i) $s(X) = w(X)$;
- (ii) for any closed subspace Y of X , $d(Y) = w(Y)$.

All metrizable spaces are in P . More generally, every regular T_1 space with a point countable base satisfying condition (i) belongs to P . Furthermore, if $F = \text{cl } F \subseteq X \in P$ and $s(F) = w(F)$, then $F \in P$ (we will use this fact without explicit mention in Cases I and II of our Theorem, below). Notice that

Lemma. If $X \in P$ and $A \subseteq X$, then $|\text{cl } A| \leq |A|^{t(X)}$.

Proof. If $B \subseteq X$ and $|B| \leq t(X)$, then by (ii), $w(\text{cl } B) \leq t(X)$. Hence $|\text{cl } B| \leq 2^{t(X)}$. Since $\text{cl } A = \bigcup \{\text{cl } B : B \subseteq A \text{ and } |B| \leq t(X)\}$, it follows that

$$|\text{cl } A| \leq 2^{t(X)} \cdot |A|^{t(X)} = |A|^{t(X)}. \quad \square$$

Let (S) denote the following statement:

(S): If X is a topological space such that $|X| > 2^{t(X)}$ and $\text{cf}(w(X)) > t(X)$, then $|X| = w(X)$.

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Theorem. *The following are equivalent:*

- (a) SCH.
- (b) If $X \in P$, then X satisfies (S).
- (c) If X is a metrizable space, then X satisfies (S).

Proof. (a) $\rightarrow$ (b). Let $X \in P$ be such that $|X| > 2^{t(X)}$ and $\text{cf}(w(X)) > t(X)$. Let $\tau = t(X)$ and $\lambda = w(X)$. Let $D = \{x_\alpha : \alpha < \lambda\}$ be a dense subset of X . By the above lemma, $|X| \leq \lambda^\tau$. Since $|X| > 2^\tau$, $\lambda > 2^\tau$. For each $\beta < \lambda$, let $Z_\beta = \text{cl}\{x_\alpha : \alpha < \beta\}$. If $x \in X$, then there exists a subset B of D such that $|B| \leq \tau$ and $x \in \text{cl} B$. Since $\text{cf}(\lambda) > \tau$, there exists $\beta < \lambda$ such that $B \subseteq \text{cl}\{x_\alpha : \alpha < \beta\}$. Thus $x \in Z_\beta$ and therefore $\bigcup\{Z_\alpha : \alpha < \lambda\} = X$. To prove that $|X| = \lambda$ it is enough to show that $|X| \leq \lambda$; we do this by induction on λ .

If $\lambda = (2^\tau)^+$, then, by the preceding lemma, $|Z_\beta| \leq 2^\tau$, for each $\beta < \lambda$. Hence $|X| \leq 2^\tau \cdot \lambda = \lambda$.

Now let $\lambda > (2^\tau)^+$, and suppose (S) has been established for all spaces in P of weight $< \lambda$.

Case I. λ is a limit cardinal. Let $\{\lambda_\alpha : \alpha < \text{cf}(\lambda)\}$ be an increasing sequence of regular cardinals converging to λ such that $\lambda_\alpha > 2^\tau$ for each $\alpha < \text{cf}(\lambda)$. By (i), for each $\alpha < \text{cf}(\lambda)$, there exists a discrete subset Y_α of X such that $|Y_\alpha| = \lambda_\alpha$. Let $X_\alpha = \text{cl}(Z_{\lambda_\alpha} \cup Y_\alpha)$, for each $\alpha < \text{cf}(\lambda)$. Then $w(X_\alpha) = \lambda_\alpha$ and $X_\alpha \in P$. Also $|X_\alpha| \geq \lambda_\alpha > 2^\tau \geq 2^{t(X_\alpha)}$ and $\text{cf}(\lambda_\alpha) = \lambda_\alpha > \tau \geq t(X_\alpha)$. Therefore, by the inductive hypothesis, $|X_\alpha| = \lambda_\alpha$ for each $\alpha < \text{cf}(\lambda)$. Since $X = \bigcup\{X_\alpha : \alpha < \text{cf}(\lambda)\}$, $|X| \leq \lambda$.

Case II. $\lambda = \kappa^+$, for some κ such that $\text{cf}(\kappa) > \tau$. By (i), there exists a discrete subset Y of X such that $|Y| = \kappa$. For each α , $\kappa < \alpha < \lambda$, let $X_\alpha = \text{cl}(Z_\alpha \cup Y)$. Then $w(X_\alpha) = \kappa$ and $X_\alpha \in P$. Since $\kappa^+ = \lambda > (2^\tau)^+$, $\kappa > 2^\tau$. Also, $|X_\alpha| \geq \kappa > 2^\tau \geq 2^{t(X_\alpha)}$ and $\text{cf}(\kappa) > \tau \geq t(X_\alpha)$. Therefore, by the inductive hypothesis, $|X_\alpha| = \kappa$ for each α . Since $X = \bigcup\{X_\alpha : \kappa < \alpha < \lambda\}$, $|X| \leq \lambda$.

Case III. $\lambda = \kappa^+$, for some κ such that $\text{cf}(\kappa) \leq \tau$. Then $2^{\text{cf}(\kappa)} \leq 2^\tau < \kappa$. Therefore, by SCH, $\kappa^{\text{cf}(\kappa)} = \kappa^+ = \lambda$. Hence $\lambda^\tau = \kappa^{\text{cf}(\kappa) \cdot \tau} = \kappa^\tau$. Again, by SCH, it follows that $\kappa^\tau = \kappa^+$ (cf. [Je, Lemma 8.1]). Thus $\lambda^\tau = \lambda$. Hence, by the preceding lemma, $|X| \leq \lambda$.

The implication (b) $\rightarrow$ (c) is obvious.

(c) $\rightarrow$ (a). In view of a theorem of Silver (cf. [Je, Theorem 23]), it is enough to show that SCH holds for singular cardinals of cofinality ω . So let κ be a singular cardinal such that $\text{cf}(\kappa) = \omega$ and $2^\omega < \kappa$. Let Y be a discrete space of cardinality κ^+ , and let X be the product of countably many copies of the space Y , $X = Y^\omega$. Then X is metrizable, $w(X) = \kappa^+$, $t(X) = \omega$, and $|X| = (\kappa^+)^\omega > 2^\omega$. Hence, by (S), $(\kappa^+)^\omega = \kappa^+$. Thus $\kappa^\omega \leq \kappa^+$. Also, by König's Theorem (cf. [Je, Theorem 17]), $\kappa^\omega \geq \kappa^+$. Hence $\kappa^\omega = \kappa^+$. $\square$

Example. Let $\{\kappa_n : n < \omega\}$ be any increasing sequence of cardinals $\geq 2^\omega$, for instance, $\kappa_0 = 2^\omega$, $\kappa_{n+1} = \kappa_n^+$ for each $n < \omega$. We set $\lambda = \sup\{\kappa_n : n < \omega\}$. Consider κ_n to be a discrete space, and let X be the product of the discrete spaces κ_n , $X = \prod\{\kappa_n : n < \omega\}$. Then X is a metrizable space (and thus $X \in P$), $w(X) = \lambda$, and $|X| = \lambda^\omega > \lambda$. Thus $w(X) < |X|$.

This example shows that the requirement in (S) that $\text{cf}(w(X)) > t(X)$ is essential. $\square$

REFERENCES

- [Je] T. Jech, *Set theory*, Academic Press, New York, 1978.
- [Ju] I. Juhász, *Cardinal functions in topology—Ten years later*, Math. Centre Tracts, Amsterdam, 1980.

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