A NEW CHARACTERIZATION OF TOTALLY PROJECTIVE GROUPS OF CARDINALITY $\aleph_1$

PAUL HILL

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ABSTRACT. A new characterization is given for totally projective groups of cardinality not exceeding $\aleph_1$. The new description is in terms of what we call a v -basis. The question about the structure of larger groups having a v -basis is left open, but we establish some closure properties of such groups.

1. INTRODUCTION

Throughout, G will denote a p -primary abelian group written additively. Recall that we define $pG = \{px : x \in G\}$, and $p^\alpha G$ is defined inductively, for any ordinal α , by the equations $p^{\alpha+1}G = p(p^\alpha G)$ and $p^\beta G = \bigcap_{\alpha < \beta} p^\alpha G$ if β is a limit. Perhaps needless to say, we define $p^0 G = G$. We let $|g|$ denote the height of an element g in G . Thus $|g| = \alpha$ means that $g \in p^\alpha G \setminus p^{\alpha+1} G$; in case $g \in p^\alpha G$ for every α , we write $|g| = \infty$. It should be observed that $|x + y| \geq \min\{|x|, |y|\}$, and equality holds unless $|x| = |y|$. In the latter case, either equality or strict inequality may prevail. For a p -primary group G , we let $G[p] = \{x \in G : px = 0\}$. Since $G[p]$ is bounded by p , it is naturally a vector space over the prime field of characteristic p . Likewise, so is $p^\alpha G[p]$, and the dimension of the quotient space $p^\alpha G[p]/p^{\alpha+1} G[p]$ is called the α^{th} Ulm invariant of G although Ulm [U], himself, did not specifically use these numerical invariants.

The concept of a totally projective group is due to R. J. Nunke [N1]. A p -primary group G is *totally projective* if $p^\alpha \text{Ext}(G/p^\alpha G, X) = 0$ for every group X and ordinal α . For the initial in depth study of totally projective groups, see [N2]. P. Crawley and A. Hales were the first to study simply presented primary groups in [CH]. A group G is *simply presented* if it can be presented by generators and relations where each relation involves at most two generators. It is demonstrated in [H1] and [H2] (and elsewhere; see, for example, [F]) that for any primary group G the following are equivalent:

- (a) G is totally projective.

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- (b) G is simply presented.
- (c) G satisfies Axiom 3 with respect to nice subgroups.

For completeness, we explain (c). A subgroup H of G is said to be *nice* if $p^\alpha(G/H) = (p^\alpha G + H)/H$ for each ordinal α . The group G satisfies Axiom 3 (with respect to nice subgroups) if G contains a collection $\mathcal{C}$ of nice subgroups for which the following conditions are satisfied:

- (0) $0 \in \mathcal{C}$.
- (1) $\mathcal{C}$ is closed with respect to group unions.
- (2) If K is countable and $N \in \mathcal{C}$, there exists $N' \in \mathcal{C}$ where N'/N is countable and $N' \supseteq N + K$.

For other equivalent versions of Axiom 3, see [H2]. One of these variations is that G has a smooth chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots$$

of nice subgroups that lead up to $G = \bigcup N_\alpha$, where $N_{\alpha+1}/N_\alpha$ is countable for each α . Being smooth means that there are no jumps at limit ordinals. In other words, $N_\beta = \bigcup_{\alpha < \beta} N_\alpha$ whenever β is a limit. Technically, each of the different versions of Axiom 3 yields a slightly different characterization of totally projective groups, but we shall count these as only one generic characterization. Even so, there are still other descriptions of totally projective groups aside from the three main characterizations listed above. For example, G is totally projective if and only if it is a balanced projective (see Theorem 81.9 in [F]).

The class of totally projective (= simply presented = Axiom 3 = balanced projective) groups is without doubt one of the most important classes of abelian groups. This is partly because these groups can be described in the various ways we have mentioned, but is mainly because these groups are completely determined and classified by their Ulm invariants [H1]. In this paper, we find another characterization for totally projective groups of cardinality not exceeding $\aleph_1$. This is in terms of the existence of a certain kind of basis which is called a *v*-basis. The structure of larger groups, however, seems more complicated, and we leave that question open.

2. *v*-BASIS

Let G be any p -primary abelian group.

Definition 1. For each ordinal λ , let B_λ be a set of representatives of the nonzero cosets of $p^\lambda G$ mod $p^{\lambda+1}G$; in other words, B_λ contains exactly one element from each of the nonzero cosets of $p^{\lambda+1}G$ in $p^\lambda G$. If each element g in G can be written as

$$(R) \quad g = b_1 + b_2 + \cdots + b_n,$$

where $b_i \in B_{\lambda(i)}$ with $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$, then $B = \bigcup B_\lambda$ is called a *v*-basis of G . (It is understood that the vacuous sum represents zero.) The expression (R) is called a representation of g with respect to the *v*-basis B .

Remark. In order to accommodate groups that are not reduced, we can adjoin ∞ to the class of ordinals with the understanding that $\lambda < \infty$ for every ordinal λ . It is convenient here, however, *not* to allow the usual $\infty < \infty$ in regard to the definition of B_∞ . We, in fact, define B_∞ to be the nonzero elements of $p^\infty G = \bigcap p^\alpha G$, where α ranges over all ordinals.

Lemma 1. *If $B = \bigcup B_\lambda$ is a v -basis of G , then the representation (R) of each element g in G with respect to this basis is unique.*

Proof. Let $g = b_1 + b_2 + \cdots + b_n = b'_1 + b'_2 + \cdots + b'_m$ be two representations of g with respect to the v -basis $B = \bigcup B_\lambda$. Hence, suppose that $b_i \in B_{\lambda(i)}$ and $b'_j \in B_{\mu(j)}$, where $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$ and $\mu(1) < \mu(2) < \cdots < \mu(m)$. Since $\lambda(1) = |g| = \mu(1)$, we quickly conclude that $\lambda(1) = \mu(1)$ and that $b_1 = b'_1$. If we replace g by $g - b_1 = g - b'_1$, the proof then follows by induction on n .

The following example demonstrates that not all primary groups have a v -basis.

Example 1. Let G be the torsion completion of an unbounded, countable p -group having no elements of infinite height. More specifically, we can take G to be the torsion product of the cyclic groups $C(p^n)$. Let $C_n = \langle c_n \rangle$ be a cyclic group of order p^n , and let G be the torsion subgroup of the product $\prod C_n$. As is well known, in the specific case we are now considering or in the more general case which we began, G has cardinality of the continuum and $p^\omega G = 0$. Moreover, for each nonnegative integer n , $p^n G / p^{n+1} G$ is countable. Now, suppose that G has a v -basis $B = \bigcup B_n$. Since $p^n G / p^{n+1} G$ is countable, B_n must be countable. Therefore, $B = \bigcup B_n$ is countable, and consequently G must be countable. But this is not the case, and we conclude that G cannot have a v -basis.

The next lemma shows that the class of groups having a v -basis is closed with respect to direct sums.

Lemma 2. *If $G = \bigoplus_{i \in I} G_i$ is a direct sum of groups G_i that have v -bases, then G itself has a v -basis.*

Proof. Let $B_i = \bigcup B_{i,\lambda}$ be a v -basis for G_i . Then $B_{i,\lambda}$ is a set of representatives for the nonzero cosets of $p^{\lambda+1} G_i$ in $p^\lambda G_i$, and each element $g_i \in \mathcal{I}_i$ has a unique representation

$$g_i = b_{i,1} + b_{i,2} + \cdots + b_{i,n},$$

where $b_{i,j} \in B_{i,\lambda(j)}$ with $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$.

We define $B_\lambda = \bigoplus_{i \in I} B_{i,\lambda}$ ($= \{\sum_i b_{i,\lambda}\}$, where the sum is finite but not vacuous, and $b_{i,\lambda} \in B_{i,\lambda}$) and let $B = \bigcup B_\lambda$. Our claim is that B is a v -basis for G . To verify this, first observe that B_λ is a set of representatives of the nonzero elements of $p^{\lambda+1} G$ in $p^\lambda G$. This is an immediate consequence of the fact that $p^\lambda G = \bigoplus_{i \in I} p^\lambda G_i$. It remains only to show that each element g in G can be written as

$$g = b_1 + b_2 + \cdots + b_n,$$

where $b_j \in B_{\lambda(j)} = \bigoplus_{i \in I} B_{i,\lambda(j)}$ with $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$. But we know that $g = \sum g_i$, where the sum is finite and $g_i \in G_i$. Moreover, since $B_i = \bigcup B_{i,\lambda}$ is a v -basis of G_i , we know that $g_i = \sum b_{i,j}$, where $b_{i,j} \in B_{i,\lambda(j)}$ with $\lambda_i(1) < \lambda_i(2) < \cdots$. Define $\Lambda = \{\lambda: \lambda = \lambda_i(j) \text{ for some } i \text{ and } j\}$, and list this finite set in increasing order $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$. For each $\lambda(k) \in \Lambda$, we set $b_k = \sum b_{i,j}$, where $\lambda_i(j) = \lambda(k)$. Then $b = b_1 + b_2 + \cdots + b_n$, where $b_k \in B_{\lambda(k)} = \bigoplus_i B_{i,\lambda(k)}$, and B is a v -basis of G .

The next lemma can be interpreted to mean that any group that has a basis also has a v -basis (which is different from the basis). The proof is trivial.

Lemma 3. Let $A = \bigoplus_{n < \omega} A_n$, where $A_n = \bigoplus \langle a_{n,i} \rangle$ is a direct sum of cyclic groups of order p^n . Define $B_k = \{ \sum_{n > k} t_{n,i} p^k a_{n,i} : 0 \leq t_{n,i} \leq p-1 \}$, where the sum is finite but not trivial. Then $B = \bigcup_{k < \omega} B_k$ is a v -basis of A . In particular, any bounded p -group has a v -basis.

Perhaps we should insert a word of caution here. The fact that G has a v -basis does not imply that $B = \bigcup B_\lambda$ is a v -basis of G for every choice B_λ of representatives of $p^\lambda G \bmod p^{\lambda+1} G$. The following simple example illustrates this fact.

Example 2. Let $G = \bigoplus_{n \geq 1} \langle c_n \rangle$ be a direct sum of cyclic groups where the order of c_n is p^n for each positive integer n . As in Lemma 3, $B_n = \{ \sum_{i > n} t_i p^n c_i : 0 \leq t_i \leq p-1 \}$ is a set of representatives of $p^n G \bmod p^{n+1} G$ for all $n < \omega$. Thus $B = \bigcup B_n$ is a v -basis of G . Now, let $H = \bigoplus \langle c_n - p c_{n+1} \rangle$. Then H is a pure subgroup of G with the property that $G/H = Z(p^\infty)$ is divisible. Let $A = \bigcup A_n$ be a v -basis for H , where A_n is a set of representatives of $p^n H \bmod p^{n+1} H$. Since $p^n G = p^n H + p^{n+1} G$, clearly A_n is also a set of representatives of $p^n G \bmod p^{n+1} G$. But obviously A cannot be a v -basis of G since $A \subseteq H \neq G$.

The next lemma will prove to be a major component in the proof of our first major result, Theorem 1.

Lemma 4. Let G be a p -primary group and γ an ordinal. If $p^\gamma G$ and $G/p^\gamma G$ both have a v -basis, then G has a v -basis.

Proof. For $\lambda < \gamma$, let $\bar{A}_\lambda = A_\lambda / p^\gamma G$ be a set of representatives of $p^\lambda (G/p^\gamma G) = p^\lambda G / p^\gamma G \bmod p^{\lambda+1} (G/p^\gamma G) = p^{\lambda+1} G / p^\gamma G$, where $A_\lambda \subseteq p^\lambda G$. Certainly, A_λ is a set of representatives of $p^\lambda G \bmod p^{\lambda+1} G$. Let C_λ be a set of representatives of $p^\lambda (p^\gamma G) = p^{\gamma+\lambda} G \bmod p^{\lambda+1} (p^\gamma G) = p^{\gamma+\lambda+1} G$. Define $B_\lambda = A_\lambda$ if $\lambda < \gamma$ and let $B_{\gamma+\lambda} = C_\lambda$. In either case, B_μ is a set of representatives of $p^\mu G \bmod p^{\mu+1} G$. Moreover, if we choose A_λ and C_λ so that $\bar{A} = \bigcup \bar{A}_\lambda$ and $C = \bigcup C_\lambda$ are v -bases of $G/p^\gamma G$ and $p^\gamma G$, respectively, then B is a v -basis of G .

Corollary 1. If G is a p -primary group and pG has a v -basis, then so does G .

Through the preceding series of lemmas, we have established the essentials for the proof of the following.

Theorem 1. Any totally projective group has a v -basis.

Proof. Let G be totally projective. Since any divisible group G has v -basis $B = B_\infty = G \setminus 0$, we may assume, by virtue of Lemma 2, that G is reduced. Let G be reduced of length σ , that is, let σ be the smallest ordinal for which $p^\sigma G = 0$. The proof is by induction on σ . If $\sigma = 1$, in fact if $\sigma = n < \omega$, then G has a v -basis according to Lemma 3. If σ is infinite, there are two cases.

Case 1: σ is a limit. As was first proved by Nunke [N2], $G = \bigoplus G_i$ where G_i is a totally projective group of smaller length than σ . By the induction hypothesis G_i has a v -basis. Hence, again by Lemma 2, G has a v -basis.

Case 2: σ is isolated. Then $p^{\sigma-1} G$ being bounded by p has a v -basis according to Lemma 3. Moreover, $G/p^{\sigma-1} G$ being a totally projective group of length $\sigma-1 < \sigma$ has a v -basis according to the induction hypothesis. Finally,

Lemma 4 implies that G has a v -basis, which completes the proof of the theorem.

3. SECURE SUBGROUPS

Suppose that G is a primary group with a v -basis $B = \bigcup B_\lambda$. Recall that if $g \in G$, then

$$(R) \quad g = b_1 + b_2 + \cdots + b_n,$$

where $b_i \in B_{\lambda(i)}$ with $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$, and this (unique) expression is called a representation of g with respect to the v -basis B .

Definition 2. A subgroup H of a group G with a v -basis B is called a *secure* subgroup (with respect to B) if H satisfies the following condition. If $h \in H$ and $h = b_1 + b_2 + \cdots + b_n$ is the representation of h , then $b_i \in H$ for each i .

The next lemma is crucial for our new characterization of totally projective groups of cardinality not exceeding $\aleph_1$.

Lemma 5. *Every secure subgroup is nice.*

Proof. Let G be a group with a v -basis $B = \bigcup B_\lambda$ and let H be a secure subgroup of G with respect to B . Suppose that $g \in G \setminus H$. In order to show that H is nice in G , it suffices to show that the coset $g + H$ has a proper element g_0 , that is, to show that there is an element $g_0 \in g + H$ with the property that $|g_0| \geq |g_0 + h|$ for all $h \in H$. In order to do this, among all the elements in the coset $g + H$, choose g_0 to have the shortest possible representation

$$g_0 = b_1 + b_2 + \cdots + b_n.$$

Let $b_i \in B_{\lambda(i)}$ where $\lambda(1) < \lambda(2) < \cdots < \lambda(n)$. Using the fact that n is minimal, we will show that g_0 is proper. Suppose that g_0 is not proper, and let $|g_0 - h| < |g_0|$ where $h \in H$. Let

$$h = b'_1 + b'_2 + \cdots + b'_m$$

where $b'_i \in B_{\mu(i)}$ with $\mu(1) < \mu(2) < \cdots < \mu(m)$. Since $|g_0 - h| < |g_0|$, it follows that $|h| = |g_0|$ and therefore $\mu(1) = |h| = |g_0| = \lambda(1)$. This implies that $b_1 = b'_1$. But H is secure, so $b_1 = b'_1 \in H$. This, however, yields a contradiction since $g_0 - b_1 \in g + H$ has a shorter representation than g_0 . Therefore, H is nice in G , and the lemma is proved.

We now come to our main result.

Theorem 2. *Let G be a primary abelian group of cardinality not exceeding $\aleph_1$. Then G is totally projective if and only if G has a v -basis.*

Proof. If G is totally projective, it has a v -basis by Theorem 1. Conversely, suppose that G has v -basis $B = \bigcup B_\lambda$. In the introduction, we mentioned that one version of Axiom 3 is that G has a smooth chain of nice subgroups

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots$$

that leads up to $G = \bigcup N_\alpha$ with the property that $N_{\alpha+1}/N_\alpha$ is countable for each α . Therefore, the theorem will be proved if we can show that there exists such a chain of nice subgroups. If G is countable, there is nothing more to prove.

Hence, assume that $|G| = \aleph_1$. In view of Lemma 5, we need only establish the desired chain $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots$ of secure subgroups. The main advantage of considering secure subgroups is that the property of being a secure subgroup, unlike that of being a nice subgroup, is inductive; all secure subgroups here are understood to be with respect to the fixed v -basis B of G . Since $|G| = \aleph_1$, it now suffices to show that any countable subgroup K of G is contained in a countable secure subgroup H of G .

Let K be any countable subgroup of G . Set $H_0 = K$. We define H_n inductively as follows:

$$H_{n+1} = \langle b_i : x = b_1 + b_2 + \cdots + b_n, \text{ where } x \in H_n \rangle.$$

It should be understood in the preceding defining equation of H_{n+1} that $b_1 + b_2 + \cdots + b_n$ is the representation of x with respect to the v -basis B of G . Finally, we set $H = \bigcup_{n < \omega} H_n$. Since H is obviously secure and remains countable, the proof is finished.

It remains to investigate groups of cardinality larger than $\aleph_1$ that have v -bases. We have not been able to modify the proof of Theorem 2 so that it applies, but perhaps some other approach might work.

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, ALABAMA 36849-5310