INVARIANT THEORY FOR A PARABOLIC SUBGROUP
OF $\mathrm{SL}(n+1, \mathbb{R})$

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ABSTRACT. For a certain maximal parabolic P of $\mathrm{SL}(n+1, \mathbb{R})$, the complete invariant theory is presented for a class of P-representation modules. These modules arise naturally from the geometry of $\mathbb{P}^n$. In particular, a means of listing all the exceptional invariants is described. This is a model problem for some parabolic invariant theory problems posed by Fefferman.

1. INTRODUCTION

For his work on pseudoconvex domains in $\mathbb{C}^n$, Fefferman was drawn to consider the problem of listing all the scalar invariants for CR structures. This requires an invariant theory for certain parabolic subgroups of a semisimple Lie group analogous to Weyl's invariant theory for classical groups. Of course, there are similar interesting problems for other parabolics and Fefferman considered one such as a model for his original question; it turns out that this algebraic problem arises from an interesting problem in conformal geometry (see [EG]). Fefferman was partially successful in his endeavour [F] but he ran against serious technical difficulties.

Since then, the scope of the investigation has been broadened to include the parabolic invariant theory associated with a variety of geometric problems in conformal and projective geometry (see [Gr] for a general review and a discussion of the relationship between the various geometric problems and their algebraic counterparts). A detailed description of the problem we consider is given in section 2, but, in brief, it is as follows. Fix some point e_0 of $\mathbb{R}^{n+1}$ and let P be the parabolic subgoup of $\mathrm{SL}(n+1, \mathbb{R})$ which stabilises the ray through e_0. The jets, at e_0, of $\mathbb{R}^{n+1}$-valued divergence-free homogeneous functions which vanish to some fixed order form a P-module and the object is to find the invariants of this module, that is, the scalar-valued polynomials in these jets

which simply scale under the *P*-action. These modules and invariants can also be viewed as arising in connection with the jets of vector fields on $\mathbb{P}^n$ (or, alternatively, on S^n as the space of rays in $\mathbb{R}^{n+1}$). This geometrical interpretation is discussed briefly in section 3.1. However for the purposes of our construction the *P*-modules are best described algebraically in terms of tensors, which are essentially the standard coordinate derivatives of the jets. One way to form an invariant is to juxtapose an appropriate number of these tensors against volume forms and then remove all indices by taking traces. Such invariants and linear combinations thereof are called *Weyl* invariants while invariants which cannot be constructed in this manner are said to be *exceptional* invariants. Since it is possible to list all the Weyl invariants, a primary concern is to determine to what extent all invariants are Weyl invariants.

Problems of the sort considered here are particularly important as guides for the analogous problems in the conformal and CR cases. In [Go] I gave a complete solution of the similar question where one begins with homogeneous functions (rather than vector-valued homogeneous functions). Following this and using some new ingredients, Bailey, Eastwood, and Graham [BEGr] completely solved the original question posed by Fefferman for CR geometry, his model problem, and also an analogous algebraic question related to constructing all invariants of conformal structures. It turns out that in the first of these cases there are no exceptional invariants but in the other two problems there are. Thus their results threw emphasis on the need to devise a scheme for listing the exceptional invariants. For this question the modules considered in [Go] do not provide a useful guide, as it is rather easy to list all the exceptionals in those cases. On the other hand the modules considered here have interesting classes of exceptional invariants which are closely analogous to those which arise in Fefferman's model problem and the other conformal problem. The main result of this article is a scheme for constructing all exceptional invariants for these modules (see Theorem 3.1). The techniques described below adapt to the problems associated with conformal geometry and this is the subject of an article with Toby Bailey [BGo]. They also provide a preparatory case for the algebraic problem associated with the construction of invariants on projective geometries. Since writing this article I have solved that problem (see [Go1]).

Although this is essentially an extension of the work in [Go], as mentioned above one of the intentions is, along with presenting some new results which are interesting in themselves, to provide an easy guide for the conformal case. Also I am considering only scalar invariants (i.e., invariants taking values in a one-dimensional *P*-module) whereas [Go] deals with the more general problem of constructing tensor-valued invariants. For these reasons I have closely followed the style and notation of Bailey, Eastwood, and Graham [BEGr]. To develop notation and to provide a background for the discussion of the exceptional invariants I present first the general invariant theory of the modules concerned. This material is new and is of interest as a further application of the techniques developed in [Go]; see, in particular, Theorem 2.8. Nevertheless much of this material is analogous to material in [BEGr]'s section 2 (which treats the "conformal scalar case"). Thus when the proofs required are simple adaptions of those in [BEGr] they have been omitted except when they are needed for later work.

2. Preliminaries

Let W denote $\mathbb{R}^{n+1}$ with coordinates

$$X^I = \begin{pmatrix} X^0 \\ X^i \end{pmatrix}, \qquad i = 1, \ldots, n$$

and fix a volume form $\epsilon \in \bigwedge^{n+1} W$. Let $e_0 \in W$ be the vector with coordinates

$$e_0^I = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let G denote $SL(n+1, \mathbb{R})$—we regard W as the standard representation space of G. Define the maximal parabolic subgroup $P \subset G$ by

$$P = \{ p \in G : pe_0 = \lambda e_0, \text{ for some } \lambda > 0 \},$$

so that P consists of all elements of G of the form

(1) $$\begin{pmatrix} \lambda & r_j \\ 0 & m^i{}_j \end{pmatrix} : \lambda > 0.$$

We note that a Levi factor L of P consists of all elements of the above form with $r_j = 0$. We denote by σ_w the one-dimensional representation of P or L where the element in (1) above is represented by λ^{-w}.

Let $\mathscr{H}(w)$ denote the set of jets at e_0 of functions positively homogeneous of degree w; by 'positively homogeneous' it is meant that $f(\lambda X) = \lambda^w f(X)$ for all $\lambda > 0$. (In this paper, 'jets' means infinite jets.) If f were a genuine positively homogeneous function defined near e_0, then f would be automatically defined in a positive cone around the ray through e_0. Since P preserves the ray through e_0, it follows that P acts on the space of such functions (according to $(pf)(X) = f(p^{-1}X)$). This action of P evidently descends to jets making $\mathscr{H}(w)$ into a P-module. The group G does not act in this way since it does not preserve the ray through e_0. Its Lie algebra $\mathfrak{g}$ does act, however, giving $\mathscr{H}(w)$ the structure of a $(\mathfrak{g}, P)$-module.

Evaluation at e_0 defines a homomorphism of P-modules $\mathscr{H}(w) \to \sigma_w$. We shall write this as

$$\text{Eval} : \mathscr{H}(w) \to \sigma_w.$$

We will use indices to denote tensor-valued functions and jets. For example, we write $\mathscr{H}^{IJ}(w)$ for the jets at e_0 of $\otimes^2 W$-valued functions homogeneous of degree w. A lower index represents a W^*-valued object. The tensors with both upper and lower indices can be traced or *contracted* and this is indicated by repeating indices in the manner of Einstein's summation convention, for example $S^I T_I$. Having performed such contractions, if, as in this example, no indices remain we shall say it is a *complete contraction*, otherwise we shall say it is a *partial contraction*.

We can extend the 'Eval' map to tensor quantities, so that, for example, we have $\text{Eval} : \mathscr{H}^I(w) \to W \otimes \sigma_w$. The coordinate functions X^I define a (tautological) element of $\mathscr{H}^I(1)$. Evaluation at e_0 gives a preferred element

$e \in W \otimes \sigma_1$; we write $\mathrm{Eval}(X^I) = e^I$. The coordinate derivative $\partial_I = \partial/\partial X^I$ defines a map

$$\partial_I : \mathscr{H}^{JK\ldots M}(v) \to \mathscr{H}_I{}^{JK\ldots M}(v-1).$$

The symbol $\odot$ means the symmetric tensor product. If $w \in \{0, 1, 2, \ldots\}$, then the G-module $\odot^w W^*$, may be regarded as the polynomials on W homogeneous of degree w. Thus, we have an inclusion of $(\mathfrak{g}, P)$-modules $\odot^w W^* \hookrightarrow \mathscr{H}(w)$. Then $\mathscr{T}_w := \mathscr{H}(w)/\odot^w W^*$ is a $(\mathfrak{g}, P)$-module. As a P-module this may be identified with the complement of $\odot^w W^*$ in $\mathscr{H}(w)$ given by those jets which vanish to order $w + 1$. So

$$\mathscr{H}(w) \cong \mathscr{T}_w \oplus \odot^w W^*$$

as P-modules (but not as $(\mathfrak{g}, P)$-modules).

There is an analogous P-module splitting of the $(\mathfrak{g}, P)$-submodule of $\mathscr{H}^I(w)$ defined by

$$\mathscr{H}^I(w) := \{v^I \in \mathscr{H}^I(w) \text{ such that } \partial_I v^I = 0\}.$$

This time $\mathscr{T}_w^I := \mathscr{H}^I(w)/\mathrm{tf}(\odot^w W^* \otimes W)$ where $\mathrm{tf}(\cdots)$ means the totally trace free part and the $(\mathfrak{g}, P)$-module $\mathrm{tf}(\odot^w W^* \otimes W)$ is interpreted in the obvious way. Then as P-modules

$$\mathscr{H}^I(w) \cong \mathscr{T}_w^I \oplus \mathrm{tf}(\odot^w W^* \otimes W)$$

if $\mathscr{T}_w^I$ is identified with the submodule of jets which vanish to order $w + 1$. This splitting is also apparent from the following algebraic description of $\mathscr{H}^I(w)$.

Proposition 2.1. *As P-modules,*

$$(2) \qquad \mathscr{H}^I(w) = \left\{ \begin{array}{l} (T^{(0)}, T^{(1)}, \ldots) : T^{(k)} \in \mathrm{tf}(\odot^k W^* \otimes W) \otimes \sigma_{w-k}, \\ e \lrcorner\, T^{(k+1)} = (w-k)T^{(k)} \end{array} \right\},$$

and if $w \in \{0, 1, 2, \ldots\}$, then

$$\mathscr{T}_w^I := \left\{ \begin{array}{l} (T^{(w+1)}, T^{(w+2)}, \ldots) : T^{(k)} \in \mathrm{tf}(\odot^k W^* \otimes W) \otimes \sigma_{w-k}, \\ e \lrcorner\, T^{(k+1)} = (w-k)T^{(k)} \text{ for } k > w \text{ and } e \lrcorner\, T^{(w+1)} = 0 \end{array} \right\}.$$

Proof. Given $v^I \in \mathscr{H}^I(w)$, define irreducible tensors

$$T^{(k)} \in \mathrm{tf}(\odot^k W^* \otimes W) \otimes \sigma_{w-k}$$

by

$$T^{(k)} = T^A_{\underbrace{IJ\ldots M}_{k}} := \mathrm{Eval}\left(\partial_I \partial_J \cdots \partial_M v^A \right).$$

The condition $e \lrcorner\, T^{(k+1)} = (w-k)T^{(k)}$ follows from Euler's equation for homogeneous functions. The splitting is a consequence of the vanishing of $(w-k)$ when $w = k$, which 'decouples' the strings between $T^{(w)}$ and $T^{(w+1)}$. $\square$

The problem we are addressing is to find the P-invariants of $\mathscr{T}_w^I$. By *invariant* we mean a polynomial on $\mathscr{T}_w^I$ which is homogeneous of degree d in $\mathscr{T}_w^I$ and which simply scales under the action of P. It is sufficient to consider polynomials homogeneous in this way since P acts linearly on $\mathscr{T}_w^I$ and so a general invariant polynomial consists of invariant homogeneous parts. A class

of invariants which are easy to list are those which arise as linear combinations of complete contractions:

Definition 2.2. A *Weyl invariant* is a P-invariant of $\mathscr{T}_w^I$ constructed as a linear combination of complete contractions of the form

$$(3) \qquad \text{contr}(\epsilon \otimes \cdots \otimes \epsilon \otimes T^{(k_1)} \otimes \cdots \otimes T^{(k_d)}).$$

An invariant which cannot be written as a linear combination of complete contractions in this way is called an *exceptional invariant*.

Our problem then is first to determine to what extent all invariants arise as Weyl invariants and second to devise a scheme for listing the remaining exceptional invariants. Evidently if $I : \mathscr{T}_w^I \to \sigma_q$ is a Weyl invariant then $q = \sum_{i=1}^d (w - k_i)$. Furthermore it is clear that if $w \geq 1$ then a non-trivial Weyl invariant must have $d \geq n + 1$. If $w = 0$ then $T_B^A T_A^B$ is a Weyl invariant which fails this inequality when $n \geq 2$.

We need to understand $\mathscr{T}_w^I$ as an L-module. A tensor

$$T^{(k)} \in \text{tf}(\bigodot^k W^* \otimes W)$$

has as components the quantities

$$(4) \qquad T^0_{\underbrace{0\ldots0}_{p}\underbrace{a\ldots d}_{r}} \text{ and } T^i_{\underbrace{0\ldots0}_{p}\underbrace{a\ldots d}_{r}}, \quad p + r = k,$$

which are symmetric in the r lower case indices. The various p and r split $\bigodot^k W^*$ as an L-module (although not into irreducibles). Using that each $T^{(k)}$ is trace free and the linking between the T's as described in (2) the zeros may be eliminated and we are led to the following result:

Proposition 2.3. *The tensors $u^{(k)}$ for $k > w$ defined by*

$$(5) \qquad u^i_{\underbrace{ab\ldots d}_{k}} = T^i_{\underbrace{ab\ldots d}_{k}}$$

are symmetric on their lower indices and $u^{(w+1)}$ is trace-free, but are otherwise unrestricted. They form a spanning set for $\mathscr{T}_w^I$ and split it as an L-module.

2.1. Weak Weyl invariants.
There is another way in which invariants can arise by G-tensor operations.

Definition 2.4. Suppose

$$C : \mathscr{T}_w^I \to \bigodot^\ell W \otimes \sigma_{\ell+q}$$

is a map obtained as a linear combination of partial contractions of tensor products of the tensors e, $T^{(k)}$, and ϵ. If

$$(6) \qquad C = \underbrace{e \otimes e \otimes \ldots \otimes e}_{\ell} \otimes I,$$

for some $I : \mathscr{T}_w^I \to \sigma_q$, then I is an invariant. We call an invariant of this form a *weak Weyl invariant*.

With this definition we have the following useful result:

Proposition 2.5. *Every P-invariant of $\mathscr{T}_w^I$ arises as a weak Weyl invariant.*

The following proof is brief. More detailed proofs of corresponding results may be found in [BEGr] and [Go].

Proof. A P-invariant I of $\mathscr{T}_w^I$ is, by restriction, an invariant of the reductive subgroup L of P. Thus by Weyl's theory for reductive groups it can be written as a linear combination of complete contractions, over lower case indices, of the form

$$(7) \qquad \mathrm{contr}(\tilde{\epsilon} \otimes \tilde{\epsilon} \cdots \tilde{\epsilon} \otimes u^{(k_1)} \otimes u^{(k_2)} \cdots u^{(k_d)})$$

where $\tilde{\epsilon}$ denotes the $\mathbb{R}^n$ volume form. (Of course the dual volume form could also be used but each of these can be removed with a matching volume form in favour of traces. Thus we shall always assume it is not used.) Supposing then that I is written in this form, make the following formal changes. Replace the u's with the corresponding components of the T's using (5) and replace each $\tilde{\epsilon}^{fg\cdots i}$ with $\epsilon^{0fg\cdots i}$. Now replace the lower case traces by upper case traces and zero components using the identity

$$\psi_a^a = \psi_A^A - \psi_0^0$$

which holds for any tensor $\psi \in W \otimes W^*$. Next eliminate all lower zeros using the linking condition $e \lrcorner T^{(k+1)} = (w - k)T^{(k)}$ satisfied by the T's. Thus our invariant I is now written as a linear combination of complete contractions over the upper case indices of the quantities

$$T_{AB\cdots D}^0, \quad T_{AB\cdots D}^I, \quad \text{and} \quad \epsilon^{0FG\cdots I}.$$

Let ℓ be the maximum number of superscript 0's that occur in any summand of this formula. Observe that there exists a map

$$(8) \qquad C : \mathscr{T}_w^I \to \odot^\ell W \otimes \sigma_{\ell+q}$$

given by a linear combination of partial contractions of the tensors $T^{(k)}$ and ϵ and e such that

$$(9) \qquad C^{00\cdots 0} = I.$$

To see this note that $e^0 = 1$ and so any tensor ψ may be regarded as the $J = 0$ component of the tensor $e^J \psi$ of one higher rank. Thus, if we replace each superscript 0 in our expression by a free index and then juxtapose, against each term, as many e^J's as necessary to bring the number of free indices up to ℓ, then we obtain an expression $B^{AB\cdots E}$ such that $B^{00\cdots 0} = I$. $C^{AB\cdots E} := B^{(AB\cdots E)}$ also has this property, i.e., (9) holds.

Finally we must show that the map

$$(10) \qquad C - e \otimes e \otimes e \otimes \cdots \otimes eI : \mathscr{T}_w^I \to \odot^\ell W \otimes \sigma_{l+q}$$

is the zero map. Certainly its $00\cdots 0$ component vanishes. This is its component in the L-direct summand

$$\sigma_q \hookrightarrow \odot^\ell W \otimes \sigma_{l+q}.$$

Now $\bigodot^\ell W \otimes \mathbb{C}$ is an irreducible module for the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ and $\sigma_{-\ell} \otimes \mathbb{C} \hookrightarrow \bigodot^\ell W \otimes \mathbb{C}$ is its highest weight space. Thus every non-zero vector $\bigodot^\ell W \otimes \mathbb{C}$ may be raised to a non-zero vector in $\sigma_{-\ell} \otimes \mathbb{C}$. Now, the raising operators of $\mathfrak{g}_\mathbb{C}$ are all contained in $\mathfrak{p}_\mathbb{C}$, the complexification of the Lie algebra of P. Thus any non-zero vector in $\bigodot^\ell W \otimes \sigma_{l+q} \otimes \mathbb{C}$ yields a non-zero vector in $\sigma_q \otimes \mathbb{C}$ by consecutive application of appropriate elements of $\mathfrak{p}_\mathbb{C}$. The result now follows as the complexification of (10) is a $\mathfrak{p}_\mathbb{C}$-equivariant polynomial map. $\square$

The following proposition and its proof are used later.

Proposition 2.6. *In the expression for* I *as a weak Weyl invariant, one can take* ℓ *to satisfy*

$$\ell \leq -d - q.$$

Proof. If $C^{AB\cdots D} = e^{(A} G^{B\cdots D)}$ for some $G^{B\cdots D}$ then we can simply 'cancel' e^A from both sides of (6). Having cancelled as much as possible, we can assume that $C^{AB\cdots D}$ contains a term constructed from only the $T^{(k)}$, for $k \geq w + 1$, and ϵ. Each $T^{(k)}$ takes values in $\mathrm{tf}(\bigodot^k W^* \otimes W) \otimes \sigma_j$ for $j \leq -1$. Hence $C^{AB\cdots D}$ takes values in $\bigodot^\ell W \otimes \sigma_m$ for $m \leq -d$. On the other hand the right-hand side of (6) takes values in $\bigodot^\ell W \otimes \sigma_{q+\ell}$ and so $q + \ell \leq -d$. $\square$

2.2. Jets.

We now return to viewing elements of $\mathscr{T}_w^I$ as jets of W-valued homogeneous functions on W at e_0. We can similarly regard $\epsilon^{IJ\cdots M}$ and X^I as jets on W at e_0. Also, recall that $T^A_{IJ\cdots M} = \mathrm{Eval}(\partial_I \partial_J \cdots \partial_M v^A)$. Thus, if we substitute X^I for every occurrence of e^I in the formula for C, and replace $T^A_{IJ\cdots M}$ by $\partial_I \partial_J \cdots \partial_M v^A$ we obtain a map

$$\tilde{C} : \mathscr{T}_w^I \to \mathscr{H}\overbrace{^{AB\cdots G}}^{\ell}(q + \ell),$$

whose image lies in the symmetric part of the right-hand side and such that $\mathrm{Eval}(\tilde{C}) = C$. The relationship (6) then extends away from e_0 in the following sense.

Proposition 2.7. *There is a* $\mathfrak{g}$-*invariant mapping* $\check{I} : \mathscr{T}_w^I \to \mathscr{H}(q)$ *with* $\mathrm{Eval}(\check{I}) = I$ *such that*

$$(11) \qquad \tilde{C}\overbrace{^{AB\cdots E}}^{\ell} = X^A X^B \cdots X^E \check{I}.$$

This is an instance of Frobenius reciprocity, a general purely algebraic proof of which may be found in [Kn, Proposition 6.3]. Otherwise this proposition may be proved by an argument similar to that in [BEGr] for the corresponding conformal scalar result.

To prove the next theorem we observe that if f is a tensor homogeneous of degree u then it follows from Euler's equation that

$$(12) \qquad \partial_I(X^I f) = (n + u + 1)f.$$

Theorem 2.8. *If* $w \geq 1$ *then every non-trivial P-invariant of* $\mathscr{T}_w^I$ *has* $d \geq n$. *If* $d = n$ *then the invariant is exceptional. Otherwise the invariant arises as a*

Weyl invariant. If $w = 0$ *and* $n = 1$ *the same results hold, but if* $w = 0$ *and* $n \geq 2$ *then every P-invariant of* $\mathscr{T}_0^I$ *arises as a Weyl invariant.*

Proof. We observe, as a consequence of our being able to write an invariant in the form (7), that any P-invariant of $\mathscr{T}_w^I$, with $w \geq 1$, must have $d \geq n$. On the other hand, we observed after Definition 2.2 that when $w \geq 1$ a Weyl invariant must have $d \geq n + 1$. Thus, when $w \geq 1$, non-zero invariants with $d = n$ are exceptional invariants.

Now suppose $w \geq 1$, $d > n$ and consider the relationship

$$\tilde{C}^{AB...E} = X^A X^B \cdots X^E \tilde{I}.$$

It follows from (12) that the Weyl invariant obtained by expanding out

$$(13) \qquad\qquad \partial_A \partial_B \cdots \partial_E \tilde{C}^{AB\cdots E}$$

is some multiple of $\tilde{I}$. We must check that it is a non-zero multiple, i.e., we are not applying (12) to any homogeneity u with $n + u + 1 = 0$. The homogeneities that we are applying it to are precisely

$$q < q + 1 < \ldots < q + \ell - 1$$

and recalling that $\ell \leq -d - q$, we see that all the coefficients are non-zero provided $d > n$.

Now consider the case of $w = 0$. Recall that an invariant I can be written as a linear combination of complete contractions as in (7). Suppose that for a particular invariant only u_b^a is involved, i.e., $u^{(k)}$ where $k = 1$. Then the $\mathbb{R}^n$ volume form $\tilde{\epsilon}$ is not used in the formula for I. Thus if we replace each u_b^a formally with T_B^A then we obtain a new formula for I which is a linear combination of terms of the form of (3) and so the invariant is Weyl. On the other hand suppose that $u^{(k)}$ for $k \geq 2$ are involved. Then $\tilde{\epsilon}$ is necessarily involved in (7) and $d \geq n$. Arguing as for the above cases we see that if $d > n$ then the invariant must be Weyl whereas if $d = n$ then the invariant must be zero or exceptional. This latter case is treated below (see, in particular, the parenthetical remark in the proof of Theorem 3.1). $\square$

3. EXCEPTIONAL INVARIANTS

Let ξ_A be a jet of homogeneity -1 satisfying $X^A \xi_A = 1$; clearly such exist. Then another choice $\hat{\xi}_A$ is related to ξ_A by $\hat{\xi}_A = \xi_A + \Upsilon_A$ where $X^A \Upsilon_A = 0$.

Let $\eta^{\overbrace{AB\cdots D}^{n}} := \xi_I \epsilon^{IAB\cdots D}$. Then under $\xi_A \mapsto \hat{\xi}_A$ the change in η is of the form

$$(14) \qquad\qquad \eta^{AB\cdots D} \mapsto \hat{\eta}^{AB\cdots D} = \eta^{AB\cdots D} + X^{[A}\gamma^{B\cdots D]}.$$

Recall that, in the expression for an invariant as a linear combination of complete contractions of the form (7), if $w \geq 1$ then $\tilde{\epsilon}$ is necessarily involved. The case when $\tilde{\epsilon}$ is not involved and $w = 0$ was treated above. So we assume throughout this section that all invariants involve $\tilde{\epsilon}$ in this sense. There are

two types of exceptional invariants that we can construct using $\eta^{AB\cdots D}$. Let

$$\tilde{T}^{(w+1)} = \tilde{T}^I_{\underbrace{AB\cdots D}_{w+1}} := \partial_A \partial_B \cdots \partial_D v^I.$$

Let E_1 be the evaluation at e_0 of a linear combination of complete contractions of the form

$$\mathrm{contr}(\underbrace{\eta \otimes \cdots \otimes \eta}_{w} \otimes \underbrace{\tilde{T}^{(w+1)} \otimes \cdots \otimes \tilde{T}^{(w+1)}}_{n}).$$

Then, since $X \lrcorner \tilde{T}^{(w+1)} = 0$ it follows that E_1 is unchanged by $\xi_A \mapsto \hat{\xi}_A$ and so, when $w \geq 1$, is an exceptional invariant or is zero. Note that if $n = 1$ the invariants so constructed are zero. Observe also that when $w = 0$ we can form invariants of type E_1, but these are *not* exceptional invariants.

Suppose now one forms $\tilde{L}^{\overbrace{AB\cdots C}^{n}} = \tilde{L}^{(AB\cdots C)}$ by contracting the $n(w+1)$ lower indices of $\otimes^n \tilde{T}^{(w+1)}$ into $w+1$ $\eta^{AB\cdots D}$'s and then symmetrising over the free superscript indices. Note that $\tilde{L}$ is unchanged under $\xi_A \mapsto \hat{\xi}_A$ and therefore so is

$$(15) \qquad\qquad\qquad \partial_A \partial_B \cdots \partial_C \tilde{L}^{AB\cdots C}.$$

Let E_2 be the evaluation of this at e_0. Then, if non-zero, E_2 is an exceptional invariant. Note that if $n \geq 2$ and w is even then E_2 vanishes. Observe also that for each w there is at most one (up to scale) invariant of type E_2.

Theorem 3.1. *In dimension $n = 1$ all exceptional invariants are of type E_2. In other dimensions exceptional invariants for w even are of type E_1. For each odd w there is at most one (up to scale) exceptional invariant of type E_2 and all other exceptional invariants are of type E_1.*

Proof. Recall the proof of Theorem 2.8 and notice that if the weak form $C^{AB\cdots D}$ of an invariant has $\ell \leq -n - q - 1$ then the invariant is Weyl. Now recall from the proof of Proposition 2.6 that we can assume C contains a term constructed from only the $T^{(k)}$ (for $k \geq w+1$) and ϵ; without loss of generality we will assume such terms are symmetric. There may be more than one such term. Let us refer to such terms (and the corresponding terms in $\tilde{C}$) as the *leading terms*. Suppose that one of the leading terms has at least one $T^{(w+j)}$ where $j \geq 2$. Then $C^{AB\cdots D} \in \odot^\ell W \otimes \sigma_m$ where $m \leq -d - 1$. Thus, arguing as in that proof we now get that $\ell \leq -n - q - 1$ and so the invariant is Weyl. But $d = n$ and so if the invariant is non-zero this is impossible and so the leading terms are constructed from just $T^{(w+1)}$'s and ϵ's.

Now suppose $n \geq 2$. Consider a leading term of C. Each ϵ has $n + 1$ indices. Since there are n T's, at most n of these are contracted into T's. On the other hand at least n of these are contracted into T's since the leading term is symmetric in its free indices. Since $\otimes^n T^{(w+1)}$ has $n(w+1)$ lower indices it follows that either there are w ϵ's or there are $(w+1)$ ϵ's (and w is odd). (Thus, there are no exceptional invariants when $w = 0$ and $n \geq 2$ as claimed in Theorem 2.8.)

Consider an exceptional invariant with w ϵ's. Recall the equation (6) again. Since (at e_0) $\epsilon^{IAB\cdots D} = (n+1)e^{[I}\eta^{AB\cdots D]}$ and $e \lrcorner T^{(w+1)} = 0$ it follows

that each leading term on the left-hand side of (6) is of the form $(\bigotimes^w e) \otimes E_1$ where E_1 is an invariant as defined above. Thus these leading terms may be subtracted from both sides to yield a new C which must be either zero or the weak form of another exceptional invariant. But if the latter then, removing e's as usual, C would require leading terms. These would also each have to be of the form $(\bigotimes^w e) \otimes E_1$, which is impossible. So the new C is zero and the invariant is of type E_1.

Now consider an exceptional invariant with $(w + 1)$ ϵ's. Up to scale there is then only one way to form a leading term for $\tilde{C}$, so we assume there is just one leading term. Since $\epsilon^{IAB\cdots D} = (n+1)X^{[I}\eta^{AB\cdots D]}$ and $X \lrcorner \tilde{T}^{(w+1)} = 0$ this leading term is necessarily a scalar multiple of

$$\underbrace{X^{(A}X^B \cdots X^{C}}_{w+1} \tilde{L}^{DE\cdots F)}.$$

Now the other terms of $\tilde{C}$ are each of form $X^{(A}W^{BC\cdots F)}$ where W is a linear combination of objects constructed from the tensors $T^{(k)}$, X, and ϵ. Thus an X may be cancelled from both sides of (11) (cf. the proof of Proposition 2.6) The 'new $\tilde{C}$' on the left-hand side then has $\ell \le -d - q - 1$ and the invariant I may be recovered by differentiating as in (13). Thus I consists of a part which is an invariant of type E_2, arising from the leading term, and possibly another part arising from the other terms. However the latter, by construction, would be Weyl if non-zero. This is impossible since $n = d$ and it follows that I is an invariant of type E_2.

The $n = 1$ case follows as for the $n \ge 2$ case above except now the leading terms associated with invariants of type E_2 can arise for w of either parity whereas invariants of type E_1 are not involved in the leading terms of any invariants since the T's are trace free. $\square$

3.1. Vectors on $\mathbb{P}^n$ and S^n.

The P-modules considered above arise in a natural geometrical manner as certain jets of vectors on $\mathbb{P}^n$. Actually it is sufficient, and more direct, to work on S^n, the space of rays in $\mathbb{R}^{n+1}$. This avoids awkwardness arising from the non-orientablity of $\mathbb{P}^n$ when n is even. Since S^n is a $2-1$ covering space of $\mathbb{P}^n$ and $G = \mathrm{SL}(n+1, \mathbb{R})$ is a covering group for the group of projective motions on $\mathbb{P}^n$ it is a straightforward exercise for the reader to adapt the discussion below to $\mathbb{P}^n$. The invariants of $\mathscr{T}_w^I$ constructed above correspond to (non-linear) invariant differential differential operators on S^n. Recall that $\mathscr{H}(w)$ denotes the jets at e_0 of functions positively homogeneous of degree w. Regard the coordinates X^A as homogeneous coordinates on S^n. These functions may be thought of as sections of a line bundle which we shall denote $\mathscr{E}(w)$. Similarly $\mathscr{H}^I(w)$ may be thought of as jets of vectors in $\mathscr{E}^a(w-1) := T\mathbb{P}^n \otimes \mathscr{E}(w-1)$ at the point $[e_0]$ in S^n determined by e_0. It follows that $\mathscr{T}_w^I$ consists of such jets modulo the kernel of an invariant linear operator.

Now the operator ∂_A determines a natural family of (affine) connections on any affine patch of S^n (for details see [Go]). Writing ∇_a to denote a member of this family one can explicitly write down the differential invariants on S^n determined by the P-invariants constructed above. We conclude with an example.

Let $n = 2$ and $w = 1$. Up to scale there is then only one exceptional invariant. This is the E_2 type invariant given by

$$\partial_E \partial_F (\tilde{T}_{AB}^E \tilde{T}_{CD}^F \eta^{AC} \eta^{BD}).$$

When expanded out it is given by a non-zero multiple of

$$\epsilon^{cd} \epsilon^{ef} ((\nabla_c \nabla_e \nabla_a v^b) \nabla_d \nabla_f \nabla_b v^a + 2 (\nabla_c \nabla_e \nabla_b \nabla_a v^a) \nabla_d \nabla_f v^b$$
$$+ (\nabla_c \nabla_e \nabla_a v^a) \nabla_d \nabla_f \nabla_b v^b)$$

and so is clearly non-zero. (Thanks to Michael Eastwood for helping check this expansion.)

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