PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 5, May 1995

ON THE RANK OF AN ELEMENT OF A FREE LIE ALGEBRA

VLADIMIR SHPILRAIN

(Communicated by Roe Goodman)

ABSTRACT. Let L be a free Lie algebra over an arbitrary field K, and let $\{x_1, \dots, x_n, \dots\}$, $n \geq 2$, be a free basis of L. We define the rank of an element u of L as the least number of free generators on which the image of u under an arbitrary automorphism of L can depend. We prove that for a homogeneous element u of degree $m \geq 2$, to have rank $n \geq 2$ is equivalent to another property which in the most interesting and important case when the algebra $L = L_n$ has a finite rank $n \geq 2$ looks as follows: an arbitrary endomorphism ϕ of L_n is an automorphism if and only if u belongs to $(\phi(L_n))^m$. This yields in particular a simple algorithm for finding the rank of a homogeneous element and also for finding a particular automorphic image of this element realizing the rank.

INTRODUCTION

Let L be a free Lie algebra over an arbitrary field K, and let $X = \{x_1, \dots, x_n, \dots\}$, $n \geq 2$, be a free basis of L. Given an arbitrary element u of L, we define the rank of u (rank u) to be the least number of free generators on which the image of u under an arbitrary automorphism of L can depend.

To describe all the elements of a given rank $n \geq 2$ is an important goal, but it looks unachievable. However, when u is a homogeneous element, it is possible to discover another nature of the rank in view of the following theorem. Before we state it, we need some notational agreements.

For any Lie algebra S, by S^m we denote the mth Lie power of S, i.e., the ideal of S generated by all Lie commutators of weight m of elements from S. If we have an element u of the free Lie algebra L and write $u = u(x_1, \dots, x_n)$, this just means that no generators x_i with $i > n$ are involved in u. On the other hand, if we say that u *depends* on some x_i, this means that having written u as a sum of associative monomials and made all possible cancellations, one has at least one monomial left that involves x_i.

All unexplained notation follows that of [1].

Received by the editors January 13, 1993 and, in revised form, July 30, 1993.
1991 *Mathematics Subject Classification*. Primary 17A50, 17B40.
Partially supported by a grant from the Israeli Planning and Budgeting Committee.

Theorem. *Let* $u(x_1, \dots, x_n) \in L$ *be a homogeneous element of degree* $m \geq 2$. *Then the following statements are equivalent*:

(a) *the element* u *has rank* $n \geq 2$;
(b) *the image of* u *under an arbitrary linear automorphism of* L *depends on at least* n *free generators*;
(c) *if* ϕ *is an endomorphism of* L *and* $u \in (\phi(L))^m$, *then* $\{x_1, \dots, x_n\} \subseteq \phi(L)$.

Remark. For a homogeneous element u, the equivalence of (a) and (b) can be easily verified; the statement (b) is included just because it naturally appears in the course of the proof. However, for an arbitrary element u of L, the statements (a) and (b) are not necessarily equivalent as a simple example shows.

In the most interesting and important case when the algebra $L = L_n$ has a finite rank $n \geq 2$, we have:

Corollary. *Let* $u(x_1, \dots, x_n) \in L_n$ *be a homogeneous element of degree* $m \geq 2$. *Then* u *has rank* n *if and only if it has the following property*: *whenever* $u \in (\phi(L_n))^m$ *for some endomorphism* ϕ *of* L_n, *one has* $\phi \in \operatorname{Aut} L_n$.

In the course of the proof of our main theorem, we elaborate an algorithm for finding the rank of a homogeneous Lie element u and also for finding a particular automorphic image of u realizing this rank.

1. Preliminaries

In order to proceed with the proof of our Theorem, we have to introduce some more notation. By $U(L)$ we denote the universal enveloping algebra of L, i.e., the free associative algebra over the field K with the same set X of free generators.

There is an augmentation homomorphism $\varepsilon \colon U(L) \to U(L)$ defined by $\varepsilon(x_i) = 0$, $i = 1, 2, \dots$. The kernel of this homomorphism, which is called the augmentation ideal of $U(L)$, we denote by Δ. Then there are mappings $d_i \colon U(L) \to U(L)$, $i = 1, 2, \dots$, satisfying the following conditions whenever a, $b \in K$, and u, $v \in U(L)$:

(1) $d_i(x_j) = \delta_{ij}$;
(2) $d_i(au + bv) = ad_i(u) + bd_i(v)$;
(3) $d_i(uv) = ud_i(v) + v^\varepsilon d_i(u)$.

It follows immediately that $d_i(a) = 0$ for any $a \in K$.

We will call these mappings Fox derivations in honor of R. Fox who gave a detailed exposition of similar mappings and their properties in the case of free group algebras (see [4]). These derivations have another nature as well. The ideal Δ is a free left $U(L)$-module with a free basis $\{x_1, x_2, \dots\}$, and the mappings d_i are projections to the corresponding free cyclic direct summands. Thus any element u of Δ can be uniquely written in the form $u = \sum d_i(u)x_i$.

One can also define Fox derivatives of an arbitrary weight in the usual way; for instance, derivatives of weight 2 are defined by $d_{ij}(u) = d_j(d_i(u))$.

The following lemma is an immediate consequence of the definitions.

Lemma 1. *Let* S *be a subalgebra of* L, *and let* $v \in S^m$. *Then any Fox derivative of weight* $(m-1)$ *of* v *belongs to* $S \cdot U(L)$.

We will also need a couple of combinatorial properties of Lie elements:

Lemma 2. *Suppose an element u of L depends on some free generator x_i. Then, having written u as a sum of associative monomials and made all possible cancellations, one has at least one monomial left that begins with x_i.*

This follows from the properties of a special K-basis of L (Shirshov's basis); see, e.g., [1, 2.3.8].

Lemma 3 (see [5] and [6]). *Let $v_1, \dots, v_m$ and u be some elements of L. Suppose u belongs to the right ideal of $U(L)$ generated by $v_1, \dots, v_m$. Then u belongs to the Lie algebra generated by $v_1, \dots, v_m$.*

2. Proof of the Theorem

Theorem. *Let $u(x_1, \dots, x_n) \in L$ be a homogeneous element of degree $m \geq 2$. Then the following statements are equivalent:*

(a) *the element u has rank $n \geq 2$;*
(b) *the image of u under an arbitrary linear automorphism of L depends on at least n free generators;*
(c) *if ϕ is an endomorphism of L and $u \in (\phi(L))^m$, then $\{x_1, \dots, x_n\} \subseteq \phi(L)$.*

Proof. First we prove that (b) implies (c). By means of contradiction, we suppose there is a homogenous element $u = u(x_1, \dots, x_n)$ of degree m and rank n and an endomorphism ϕ of L such that $u \in (\phi(L))^m$, but some x_i, say x_1, does not belong to $\phi(L)$.

Write u as a sum of monomials:

$$u = \sum_{(m)} a_{(m)} x_{j_1} \cdots x_{j_m}, \tag{1}$$

where $a_{(m)} \in K$, (m) staying for a multi-index $(j_1, \dots, j_m)$.

We now consider the inclusion

$$u \in (\phi(L))^m \tag{2}$$

and take an arbitrary Fox derivative of weight $(m-1)$ of both sides of (2). We get then (in view of Lemma 1):

$$\sum_{j=1}^{n} b_j x_j \in \phi(L) U(L) \tag{3}$$

for some $b_j \in K$.

By Lemma 3, (3) implies that $\sum_{j=1}^n b_j x_j \in \phi(L)$. Thus taking all possible Fox derivatives of weight $(m-1)$ of both sides of (2), we arrive at a system of inclusions of the form

$$\sum_{j=1}^{n} b_j^{(i)} x_j \in \phi(L).$$

Denote $w_i = \sum_{j=1}^n b_j^{(i)} x_j$, and let $K\langle w_i \rangle$ be the K-linear span of all w_i. Were x_1 in $K\langle w_i \rangle$, we would have $x_1 \in \phi(L)$, contrary to our assumption. So, $x_1 \notin K\langle w_i \rangle$, which implies that the dimension of $K\langle w_i \rangle$ is less than n. Let $\{w_1, \dots, w_s\}$, $s < n$, be a basis of $K\langle w_i \rangle$. It can be extended to a basis $\{w_1, \dots, w_s, \dots, w_n, \dots\}$ of L.

Now consider linear automorphism α of the algebra L taking x_i to w_i, $i \geq 1$. Then $\alpha(u)$ takes the form $\sum_{j=1}^{s} c_j x_j' u_j$, where $c_j \in K$, $x_j' = w_j$, and $u_j \in L^{m-1}$. Thus $\alpha(u)$ is a sum of monomials, each of which begins with some x_i' from the set $\{x_1', \dots, x_s'\}$. But this means that $\alpha(u)$ depends only on $x_1', \dots, x_s'$ in view of Lemma 2, so u has rank less than n, contradicting the assumption.

Now we are going to prove that (c) implies (a). Suppose by means of contradiction that u has rank less than n, i.e., for some automorphism α of algebra L we have $\alpha(u) = v(x_1, \dots, x_k)$, $k < n$. Define the endomorphism ψ_k of L by taking x_i to x_i, $1 \leq i \leq k$, and x_i to 0, $i > k$. Then $\psi_k(\alpha(u)) = \alpha(u)$, so $\alpha^{-1}\psi_k\alpha(u) = u$.

Thus the endomorphism $\varphi_k = \alpha^{-1}\psi_k\alpha$ fixes u, and, in particular, $u \in (\varphi_k(L))^m$. This should imply, by the condition (c) of the Theorem, that $\{x_1, \dots, x_n\} \subseteq \varphi_k(L)$. Hence every element of L which depends only on $x_1, \dots, x_n$ is also in $\varphi_k(L)$.

Take an arbitrary element $w = w(x_1, \dots, x_n)$, and let $w = \varphi_k(s)$. Then $w = \alpha^{-1}\psi_k\alpha(s)$, so $\alpha(w) = \psi_k\alpha(s)$ which implies that $\alpha(w)$ depends on $x_1, \dots, x_k$ only. Thus the automorphism α takes any element of the subalgebra L_n generated by $x_1, \dots, x_n$ to an element of the subalgebra L_k generated by $x_1, \dots, x_k$. This implies that a free Lie algebra L_k of rank k is isomorphic to a free algebra $\alpha^{-1}(L_k)$ of rank greater than k. Indeed, first of all the algebra $\alpha^{-1}(L_k)$ is free as a subalgebra of a free Lie algebra L. Now having contained at least n free generators $x_1, \dots, x_n$ of the algebra L, the algebra $\alpha^{-1}(L_k)$ cannot have rank less than n which is greater than k. This contradiction completes the proof of the Theorem.

3. Concluding remarks

Remark 1. In the course of the proof of the Theorem, we have actually elaborated a simple algorithm for finding the rank of a homogeneous element u of L and also for finding a particular automorphic image of u realizing this rank. The algorithm is as follows. Let $u = u(x_1, \dots, x_n)$ be a homogeneous element of degree m. Write u in the form (1), and take all possible Fox derivatives of weight $(m-1)$ of u; this yields a set of K-linear combinations of $x_1, \dots, x_n$. If the K-span of this set equals $K\langle x_1, \dots, x_n\rangle$ (this is easily decidable), we deduce that the rank of u is equal to n. If not, we can find the rank k of this K-span (in the usual sense of linear algebra) and a basis. After that, proceeding as in the proof of the Theorem, we define a linear automorphism α of L such that $\alpha(u)$ depends on k free generators. This number k is now equal to the rank of u because applying our argument to the element $\alpha(u)$ (which is also homogeneous of degree m), we see the number k of generators occuring in $\alpha(u)$ cannot be reduced.

Remark 2. As we mentioned in the Introduction, for an arbitrary element u of L, the statements (a) and (b) are not necessarily equivalent:

Example. Take $u = [x_1, x_2] + [x_3, x_4, x_2]$. The image of u under an arbitrary linear automorphism clearly depends on at least three free generators, while for the following automorphism φ, $\varphi(u)$ depends on x_1 and x_2 only: $\varphi: x_1 \to x_1 - [x_3, x_4]$; $x_i \to x_i$, $i \geq 2$.

Remark 3. An important question is to what extent our Theorem remains valid for a free associative algebra. Let us take the simplest case of the free associative algebra A_2 of rank 2 generated by x_1 and x_2. We can exhibit some homogeneous elements of rank 2: $u_1 = x_1x_2$; $u_2 = x_1x_2 - x_2x_1$; $u_3 = x_1x_2 + x_2x_1$.

In [3], it was proved that whenever $\varphi(u_2) = au_2$ for some endomorphism φ of A_2 and some nonzero element a of the main field K, this φ is actually an automorphism of A_2 (the converse is also true: if φ is an automorphism of A_2, then $\varphi(u_2) = au_2$; see [2]). The question is whether or not this remains valid when replacing the condition "$\varphi(u_2) = au_2$" by a weaker condition (c) of our Theorem: "$u_2 \in (\varphi(A_2))^2$". I believe it is so. The same question for the elements u_1 and u_3 (especially for u_1) is also interesting.

References

1. Yu. A. Bahturin, *Identical relations in Lie algebras*, VNU Science Press, Utrecht, 1987.
2. A. J. Czerniakiewicz, *Automorphisms of free algebras of rank two*, II, Trans. Amer. Math. Soc. **171** (1972), 309–315.
3. W. Dicks, *A commutator test for two elements to generate the free algebra of rank two*, Bull. London Math. Soc. **14** (1982), 48–51.
4. R. H. Fox, *Free differential calculus.* I. *Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560.
5. C. Reutenauer, *Applications of a noncommutative Jacobian matrix*, J. Pure Appl. Algebra **77** (1992), 169–181.
6. V. Shpilrain, *On generators of L/R^2 Lie algebras*, Proc. Amer. Math. Soc. **119** (1993), 1039–1043.

Department of Mathematics, Technion, Haifa 32000, Israel
Current address: Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany
E-mail address: `vladimir.shpilrain@ruba.rz.ruhr-uni-bochum.de`