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K-GROUPS OF SOLENOIDAL ALGEBRAS I

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ABSTRACT. Multiplication by x determines an automorphism of the compact dual group of $\Lambda_g = \mathbb{Z}[x, x^{-1}]/(g)$ for $g \in \mathbb{Z}[x]$. We determine the K-groups of the C^*-algebra associated with this dynamical system if g is irreducible and has degree one or two. Partial results are included if the degree of g is three.

To each $g \in \mathbb{Z}[x]$ associate a C^*-algebra B_g, the crossed product C^*-algebra associated to the dynamical system $(\widehat{\Lambda}_g, \alpha, \mathbb{Z})$. Here $\widehat{\Lambda}_g$ is the dual group of the discrete abelian group $\Lambda_g = \mathbb{Z}[x, x^{-1}]/(g)$ where (g) is the principal ideal generated by g in the ring $\mathbb{Z}[x, x^{-1}]$. The automorphism α of $\widehat{\Lambda}_g$ is that defined by multiplication by x on Λ_g. The dynamical systems $(\widehat{\Lambda}_g, \alpha)$ are examples of (abelian) Markov groups ([8]), in particular they are generalized solenoids ([2]).

In this note the K-groups of B_g are computed for g nonconstant, irreducible, and of degree one or two. Partial results for g degree three are also obtained. This is accomplished by a straightforward (though involved) application of the Pimsner-Voiculescu six-term exact sequence ([7]). In addition, the range of any state on $K_0(B_g)$ arising from a tracial state on B_g is shown to be $\mathbb{Z}$ for any (nonconstant, irreducible) g.

Although of little interest in their own right, these calculations do allow a comparison of the computed K-groups with the known (for degree g equal to one or two) $*$ (or anti-$*$)-isomorphism classes of these algebras [1]. This yields many examples of non-$*$ (or anti-$*$)-isomorphic algebras with isomorphic K-groups and, since both the tracial states on K_0 and (at least in degree one) the possible order structures on K_0 provide no additional information, one is left with the interesting (especially in light of the questions raised in [3]) possibility that the K-groups are of limited value in determining the isomorphism classes of this family of amenable algebras. The isomorphism classification arrived at in [1] (for degree g one or two) used a sequence of invariants related to the entropy of the underlying dynamical system. One can contrast this with the

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family of rotation algebras, for example, where K-theoretic means provide a classification and the entropy of each underlying dynamical system is zero.

Admittedly the algebras B_g are not simple, however, they do possess a separating family of finite-dimensional quotients. Although the algebras are not AF-algebras (K_1 is nonzero), they are finite and embeddable in AF-algebras. This follows from [5], since the finite periodic points in $\widehat{\Lambda}_g$ are dense ([2], [4]) and the dynamical system $(\widehat{\Lambda}_g, \alpha)$ is thus chain recurrent.

In the following, if $g = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$ with $g(0) \neq 0$, then g^0 denotes the polynomial $\sum_{i=0}^d a_{d-i} x^i$ and $\deg(g)$ denotes the degree of g. The content of g is written $\operatorname{cont}(g)$. For $n, m \in \mathbb{Z}$, $\{n, m\}$ is the least common multiple of n and m and (n, m) is the greatest common divisor of n and m. If φ is a $\mathbb{Z}$-module map, $\operatorname{im}\varphi$ denotes the image of φ, $\ker\varphi$ denotes the kernel of φ and $\operatorname{coker}\varphi$ denotes the cokernel of φ.

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We compute the K-groups of the abelian C^*-algebra $C(\widehat{\Lambda}_g)$ for irreducible nonconstant $g \in \mathbb{Z}[x]$ and prove some preliminary algebraic results.

For $g \in \mathbb{Z}[x]$ nonconstant, irreducible let $a \in \mathbb{C}\backslash\{0\}$ denote a root of g, so $\Lambda_g \simeq \mathbb{Z}[a, a^{-1}]$. The automorphism α of $\widehat{\Lambda}_g$ is dual to the $\mathbb{Z}$-module map M_a (multiplication by a) in the ring Λ_g. Note that Λ_g is torsion free if and only if $\operatorname{cont}(g) = 1$. Thus, Λ_g is a discrete, torsion free, abelian, rank d group where $d = \deg(g)$. Although the group Λ_g is not finitely generated (unless both g and g^0 are monic), Λ_g is a direct limit of its finitely generated submodules. Since any finitely generated submodule of Λ_g is torsion free (and thus free), write Λ_g as a direct limit of submodules each isomorphic to $\mathbb{Z}^d$. For example, let $\mathscr{M}_n$ be the submodule of Λ_g generated by $\{a^{-n}, \dots, 1, a, \dots, a^{d+n}\}$ (identifying Λ_g with $\mathbb{Z}[a, a^{-1}]$). Then $\mathscr{M}_n \simeq \mathbb{Z}^d$ and $\Lambda_g = \lim_{n\to\infty}(\mathscr{M}_n, i_n)$ where $i_n\colon \mathscr{M}_n \to \mathscr{M}_{n+1}$ is the natural inclusion. The group $K_*(C^*(\mathscr{M}_n))$ is isomorphic to $\bigoplus_{j=0}^d \wedge^j \mathbb{Z}^d$ (denoted by $\wedge\mathbb{Z}^d$) with K_0 corresponding to the even indices and K_1 to the odd indices. The induced map $(i_n)_*$ is $\bigoplus_{j=0}^d \wedge^j i_n$ (denoted $\wedge i_n$). Since Λ_g is discrete and abelian, it is straightforward to see that $C^*(\Lambda_g) \simeq \lim_n(C^*(\mathscr{M}_n))$. Since K_* commutes with lim, and the wedge product of $\mathbb{Z}$-modules commutes with lim, it follows that $K_*(C^*(\Lambda_g)) \simeq \bigoplus_{j=1}^d \wedge^j \Lambda_g \simeq \bigoplus_{j=1}^d \wedge^j \mathbb{Z}[a, a^{-1}]$.

Proposition 1.1. *The automorphism α of $C(\widehat{\Lambda}_g)$ induces the map $\bigoplus_{j=0}^d \wedge^j M_a$ on $K_*(C^*(\Lambda_g))$.*

Proof. Let $\mathscr{M}_n$ be the submodules of Λ_g defined above and note that $M_a(\mathscr{M}_n) \subseteq \mathscr{M}_{n+1}$. The maps $M_n = M_a|_{\mathscr{M}_n}^{\mathscr{M}_{n+1}}$ define a homomorphism of the directed system $(\mathscr{M}_n, i_n)$ to $(\mathscr{M}_{n+1}, i_{n+1})$ and the homomorphism $\lim M_n$ of $\lim \mathscr{M}_n = \Lambda_g$ to $\lim \mathscr{M}_{n+1} = \Lambda_g$ is the map M_a. The corresponding maps $\widetilde{M}_n\colon C^*(\mathscr{M}_n) \to C^*(\mathscr{M}_{n+1})$ on the C^*-algebra level form a map of the directed system $(C^*(\mathscr{M}_n), \tilde{i}_n)$ to itself. The homomorphism $\lim \widetilde{M}_n$ of $\lim C^*(\mathscr{M}_n) = C^*(\Lambda_g)$ to itself is the map α.

The functor K_* commutes with direct limits, so the endomorphism α_* of the group $K_*(C^*(\Lambda_g))$ is the homomorphism $(\widetilde{M}_n)_*$ of the directed system

$(K_*(C^*(\mathscr{M}_n)), (\tilde{i}_{n*}))$ to $(K_*(C^*(\mathscr{M}_{n+1})), (\tilde{i}_{n+1})_*)$. However, $(\widetilde{M_n})_*$ is the map $\bigoplus_{j=0}^d \wedge^j M_n$. Finally, note that lim commutes with direct sums and with wedge products. □

Since Λ_g is a limit of $\mathbb{Z}$-submodules isomorphic to $\mathbb{Z}^d$, $\wedge^d \Lambda_g$ is isomorphic to a limit of $\mathbb{Z}$-submodules isomorphic to $\mathbb{Z}$. In particular, the elements $a^{j_0} \wedge \cdots \wedge a^{j_{d-1}}$ (with $j_k \in \mathbb{Z}$ and $j_0 < \cdots < j_{d-1}$) generate $\wedge^d \Lambda_g$ as a $\mathbb{Z}$-module. Note also that $\wedge^d \Lambda_g$ is a limit of torsion free modules, so is torsion free.

Lemma 1.2. *Let $g = \sum_{i=0}^d a_i x^i$ (nonconstant, irreducible) and $l = \{a_0, a_d\}$. If e is the element $1 \wedge a \wedge \cdots \wedge a^{d-1}$ of $\wedge^d \Lambda_g$, then $l^{-1} e \in \wedge^d \Lambda_g$.*

Proof. Let $c_j = 1 \wedge \cdots \wedge \hat{a}^j \wedge \cdots \wedge a^d$ and $b_j = a^{-1} \wedge \cdots \wedge \hat{a}^j \wedge \cdots \wedge a^{d-1}$ for $0 \le j \le d-1$. Since $g(a) = 0$, it follows that $a_d c_j = (-1)^{d-j} a_j e$ and $a_0 b_j = (-1)^{j+1} a_{j+1} e$ for $0 \le j \le d-1$. If $r = (a_0, a_d)$, then $lr = a_0 a_d$; so $a_j a_0 r^{-1} e = (-1)^{d-j} l c_j$ and $a_{j+1} a_d r^{-1} e = (-1)^{j+1} l b_j$ for $0 \le j \le d-1$. Since $a_d a_0 r^{-1} e = le \in l \wedge^d \Lambda_g$ also, it follows that $a_j a_0 r^{-1} e$ and $a_j a_d r^{-1} e \in l \wedge^d \Lambda_g$ for $0 \le j \le d$. Since $(a_0 r^{-1}, a_d r^{-1}) = 1$, we have $a_j e \in l \wedge^d \Lambda_g$ for $0 \le j \le d$. However, $\operatorname{cont}(g) = 1$, so $(a_0, a_1, \ldots, a_d) = 1$ and $e \in l \wedge^d \Lambda_g$. The result follows since $\wedge^d \Lambda_g$ is torsion free. □

The elements $a^{j_0} \wedge \cdots \wedge a^{j_{d-1}}$ of $\wedge^d \Lambda_g$ are all contained in the $\mathbb{Z}$-module $\mathbb{Z}[a_0^{-1}, a_d^{-1}]e = \mathbb{Z}[l^{-1}]e$, and since they generate $\wedge^d \Lambda_g$ as a $\mathbb{Z}$-module, $\wedge^d \Lambda_g \subseteq \mathbb{Z}[l^{-1}]e$.

Proposition 1.3. *If g, e, l are as in the preceding lemma, then $\mathbb{Z}[l^{-1}]e = \wedge^d \Lambda_g$.*

Proof. It is enough to show $\mathbb{Z}[l^{-1}]e \subseteq \wedge^d \Lambda_g$. Since $l^{-1} e \in \wedge^d \Lambda_a$, the result will follow if $\wedge^d \Lambda_g$ is a commutative ring with unit e.

First, define a multiplication on the generators $a^{j_0} \wedge \cdots \wedge a^{j_{d-1}}$, $j_0 < \cdots < j_{d-1}$, of $\wedge^d \Lambda_g$. Using the alternating d-multilinear map Δ of $\mathbb{Q}^d$ (identified with $\mathbb{Q}[a]$) into $\mathbb{Q}$ taking the value 1 on the basis $\{1, a, \ldots, a^{d-1}\}$ of $\mathbb{Q}[a]$, we identify $\wedge^d \mathbb{Q}^d$ with $\mathbb{Q}$ (e corresponding to 1). Let $\varphi_{j_0 \cdots j_{d-1}} = \varphi$ denote the $\mathbb{Q}$-linear map of $\mathbb{Q}[a]$ determined by mapping a^k to a^{j_k}, $0 \le k \le d-1$. This map is also $\mathbb{Z}$-linear and maps Λ_g to itself. We have $a^{j_0} \wedge \cdots \wedge a^{j_{d-1}} = \wedge^d \varphi(e) = (\det \varphi)e$. Define the product of $a^{j_0} \wedge \cdots \wedge a^{j_{d-1}}$ with $a^{i_0} \wedge \cdots \wedge a^{i_{d-1}}$ as $\wedge^d(\varphi_{j_0 \cdots j_{d-1}} \circ \varphi_{i_0 \cdots i_{d_1}})(e) = \det(\varphi_{j_0 \cdots j_{d-1}} \circ \varphi_{i_0 \cdots i_{d-1}})(e) = \det(\varphi_{j_0 \cdots j_{d-1}}) \cdot \det(\varphi_{i_0 \cdots i_{d-1}})e$ and extend this linearly to a product on $\wedge^d \Lambda_g$ ($\wedge^d \Lambda_g$ is a subring of $\mathbb{Q}$). □

The proof of the above proposition shows $\wedge^d \Lambda_g$ is a ring isomorphic to $\mathbb{Z}[l^{-1}]$. The endomorphism $\wedge^d M_a$ of $\wedge^d \Lambda_g$ is multiplication by $\det(M_a) = (-1)^d a_0 a_d^{-1}$ in $\mathbb{Z}[l^{-1}]$.

Proposition 1.4. *Let $g = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$ be nonconstant, irreducible. Then $\Lambda_g/(1 - M_a)\Lambda_g \simeq \mathbb{Z}/(\sum_{i=0}^d a_i)\mathbb{Z}$.*

Proof. Let $R = \mathbb{Z}[x]/(1-x)\mathbb{Z}[x]$. Then $R \simeq \mathbb{Z}$ and $\Lambda_g/(1 - M_a)\Lambda_g \simeq \mathbb{Z}/\overline{g}\mathbb{Z}$ where $\overline{g} = \sum_{i=0}^d a_i$ is the class of g in R (cf. [2]). □

Definition. If $n \neq 0$, $l \in \mathbb{N}$, define $n : l = n(n, l^{m_0})^{-1}$ where m_0 is the maximum multiplicity of any prime dividing n. Thus, $n : l$ is formed by removing from n any prime also dividing l.

Lemma 1.5. *Let $n \neq 0$, $l \in \mathbb{N}$. Then $\mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \simeq \mathbb{Z}/t\mathbb{Z}$ where $t = n : l$.*

Proof. Let H_k be the subgroup of $Z_k \simeq \mathbb{Z}$ generated by n $(k \in \mathbb{N})$. The map $\varphi_k : Z_k \to Z_{k+1}$ given by multiplication by l defines a map $\overline{\varphi}_k : Z_k/H_k \to Z_{k+1}/H_{k+1}$. We have $\mathbb{Z}[l^{-1}] \simeq \lim(Z_k, \varphi_k)$ and $\mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \simeq \lim(Z_k/H_k, \overline{\varphi}_k)$. Each Z_k/H_k is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and $\overline{\varphi}_k$ is multiplication by l. If b is a generator of $\mathbb{Z}/n\mathbb{Z}$, then $l^m b$ has order $n(l^m, n)^{-1}$. For m large enough (m larger than the maximum multiplicity of any prime dividing n) $l^m b$ has order $t = n : l$. It follows that $\mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \simeq \lim(\mathbb{Z}/t\mathbb{Z}, m_l) \simeq \mathbb{Z}/t\mathbb{Z}$ (where m_l is multiplication by l). □

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Let $\wedge^{\mathrm{ev}}\Lambda_g$ denote $\bigoplus_{j=0}^{[d/2]} \wedge^{2j}\Lambda_g$ and $\wedge^{\mathrm{odd}}\Lambda_g$ denote $\bigoplus_{j=0}^{[d/2]} \wedge^{2j+1}\Lambda_g$. The maps $\wedge^{\mathrm{ev}}M_a$ and $\wedge^{\mathrm{odd}}M_a$ are interpreted similarly. The Pimsner-Voiculescu six-term exact sequence for the K-groups of the crossed product C^*-algebra $B_g = C^*(\Lambda_g) |\times \mathbb{Z}$ is:

$$\begin{tikzcd}
\wedge^{\mathrm{ev}}\Lambda_g \arrow[r, "1-\wedge^{\mathrm{ev}}M_a"] & \wedge^{\mathrm{ev}}\Lambda_g \arrow[r, "i_*"] & K_0(B_g) \arrow[d, "\delta_0"] \\
K_1(B_g) \arrow[u, "\delta_1"] & \wedge^{\mathrm{odd}}\Lambda_g \arrow[l, "i_*"'] & \wedge^{\mathrm{odd}}\Lambda_g \arrow[l, "1-\wedge^{\mathrm{odd}}M_a"']
\end{tikzcd}$$

The cases $\deg(g) = 1, 2$, and 3 are dealt with separately.

The case $\deg(g) = 1$. The single root a of $g(x) = a_0 + a_1 x$ is $-a_0 a_1^{-1}$ and the six-term exact sequence is:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "i_*"] & K_0(B_g) \arrow[d, "\delta_0"] \\
K_1(B_g) \arrow[u, "\delta_1"] & \Lambda_g \arrow[l, "i_*"'] & \Lambda_g \arrow[l, "1-M_a"']
\end{tikzcd}$$

If $a \neq 1$, the map $1 - M_a$ is injective (Λ_g is an integral domain) and $i_* : \mathbb{Z} \to K_0(B_g)$ is an isomorphism. To compute $K_1(B_g)$, first note that $\ker \delta_1 = \operatorname{im} i_* \simeq \operatorname{coker}(1 - M_a)$. Also, $0 \to \ker \delta_1 \to K_1(B_g) \xrightarrow{\delta_1} \mathbb{Z} \to 0$ splits since $\mathbb{Z}$ is projective. Thus, $K_1(B_g) \simeq \mathbb{Z} \oplus \ker \delta_1 \simeq \mathbb{Z} \oplus \mathbb{Z}/(a_0 + a_1)\mathbb{Z}$ by Proposition 1.4.

If $a = 1$, then $\Lambda_g = \mathbb{Z}$, $B_g = C(\mathbb{T}^2)$, $K_0(B_g) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $K_1(B_g) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proposition 2.1. *If $g = a_0 + a_1 x$ is a degree 1, irreducible polynomial in $\mathbb{Z}[x]$, then*

$$K_0(B_g) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } a_0 + a_1 = 0, \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

and

$$K_1(B_g) = \mathbb{Z} \oplus \mathbb{Z}/(a_0 + a_1)\mathbb{Z}.$$

The case $\deg(g) = 2$. Let $g(x) = \sum_{i=0}^{2} a_i x^i \in \mathbb{Z}[x]$ be irreducible with $a \in \mathbb{C}$ a root and $l = \{a_0, a_2\}$. The six-term exact sequence becomes:

$$\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z}[l^{-1}] & \xrightarrow{0\oplus M(1-a_0a_2^{-1})} & \mathbb{Z}\oplus\mathbb{Z}[l^{-1}] & \xrightarrow{i_*} & K_0(B_g) \\
\delta_1\uparrow & & & & \downarrow\delta_0 \\
K_1(B_g) & \xleftarrow{i_*} & \Lambda_g & \xleftarrow{1-M_a} & \Lambda_g
\end{array}$$

Since $a\neq 1$, the map $1-M_a$ is injective and $\operatorname{im}\delta_0=0$. Thus, i_* is surjective and $K_0(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]/(1-a_0a_2^{-1})\mathbb{Z}[l^{-1}]$. To compute $K_1(B_g)$, note that $0\to\ker\delta_1\to K_1(B_g)\to\operatorname{im}\delta_1\to 0$ is exact and multiplication by $1-a_0a_2^{-1}$ is either injective (if $a_2\neq a_0$) or zero (if $a_2=a_0$). In the first case, $\operatorname{im}\delta_1=\mathbb{Z}$; in the second case, $\operatorname{im}\delta_1=\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]$. Also note that $\ker\delta_1=\operatorname{im}i_*\simeq\operatorname{coker}(1-M_a)=\mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z}$ by Proposition 1.4. Thus, if $a_2\neq a_0$, then $\operatorname{im}\delta_1=\mathbb{Z}$ is projective and $K_1(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z}$.

If $a_2=a_0$, then $K_1(B_g)$ is an extension of $\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]$ by $\mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z}$. It is still possible to determine $K_1(B_g)$ by computing the abelian group $\operatorname{Ext}(\mathbb{Z}[l^{-1}],\mathbb{Z}/m\mathbb{Z})$ for m, l relatively prime in $\mathbb{N}$.

Proposition 2.2. *If $m, l\in\mathbb{N}$ with $(m,l)=1$, then* $\operatorname{Ext}(\mathbb{Z}[l^{-1}],\mathbb{Z}/m\mathbb{Z})=0$.

Proof. Let $\mathbb{Z}_k\simeq\mathbb{Z}$ (with unit e_k) for $k\in\mathbb{N}_0$ and D the (free) submodule of the free $\mathbb{Z}$-module $P=\bigoplus_{k\in\mathbb{N}_0}\mathbb{Z}_k$ generated by the independent set $\{h_i | h_i=e_i-le_{i+1},\ i\in\mathbb{N}_0\}$. Since $\mathbb{Z}[l^{-1}]$ is isomorphic to $\lim(\mathbb{Z}_k,\mathscr{M}_l)$ (where M_l denotes multiplication by l), we obtain the projective presentation $0\to D\xrightarrow{\mu}P\xrightarrow{\pi}\mathbb{Z}[l^{-1}]\to 0$ of $\mathbb{Z}[l^{-1}]$. Thus, $\operatorname{Ext}(\mathbb{Z}[l^{-1}],\mathbb{Z}/m\mathbb{Z})=\operatorname{coker}\mu^*$ where $\mu^*\colon\operatorname{Hom}_{\mathbb{Z}}(P,\mathbb{Z}/m\mathbb{Z})\to\operatorname{Hom}_{\mathbb{Z}}(D,\mathbb{Z}/m\mathbb{Z})$. We show μ^* is onto. Choose $\varphi=(\varphi_k)\in\operatorname{Hom}(P,\mathbb{Z}/m\mathbb{Z})\simeq\prod_{k\in\mathbb{N}_0}\operatorname{Hom}(\mathbb{Z}_k,\mathbb{Z}/m\mathbb{Z})\simeq\prod_{k\in\mathbb{N}_0}\mathbb{Z}/m\mathbb{Z}$ where $\varphi_k\in\operatorname{Hom}(\mathbb{Z}_k,\mathbb{Z}/m\mathbb{Z})$ may be identified with the element $\varphi_k(e_k)$ of $\mathbb{Z}/m\mathbb{Z}$. The image of μ^* in $\prod_{k\in\mathbb{N}_0}\operatorname{Hom}(\mathbb{Z}h_k,\mathbb{Z}/m\mathbb{Z})$ consists of $\{\psi=(\psi_k) | \psi_k=\varphi_k-l\varphi_{k+1},\ (\varphi_k)\in\prod_{k\in\mathbb{N}_0}\mathbb{Z}/m\mathbb{Z}\}$. Since l and m are relatively prime, multiplication by l maps $\mathbb{Z}/m\mathbb{Z}$ onto itself and the equations $\psi_k=\varphi_k-l\varphi_{k+1}$ can be solved for $\varphi_k\in\mathbb{Z}/m\mathbb{Z}$ given $\psi=(\psi_k)\in\operatorname{Hom}(\mathbb{Z}h_k,\mathbb{Z}/m\mathbb{Z})$. Thus, μ^* is onto and the result follows. □

Theorem 2.3. *Let $g=\sum_{i=0}^2 a_ix^i\in\mathbb{Z}[x]$ be a degree 2 irreducible polynomial and $l=\{a_0,a_2\}$. If $a_2\neq a_0$, then $K_0(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}/t\mathbb{Z}$ with $t=(a_2-a_0):l$ and $K_1(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z}$. If $a_2=a_0$, then $K_0(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]$ and $K_1(B_g)\simeq\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]\oplus\mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z}$.*

Proof. Since a_2 and $a_2^{-1}\in\mathbb{Z}[l^{-1}]$, the ideal generated by $1-a_0a_2^{-1}$ in $\mathbb{Z}[l^{-1}]$ is the same as that generated by a_2-a_0. Thus, $\mathbb{Z}[l^{-1}]/(1-a_0a_2^{-1})\mathbb{Z}[l^{-1}]\simeq\mathbb{Z}/t\mathbb{Z}$ with $t=(a_2-a_0):l$ by Lemma 1.5, and the result for K_0 follows. It remains to compute $K_1(B_g)$ when $a_2=a_0$. In this case, $l=a_0$ and $(l,\sum_{i=0}^2 a_i)=(a_0,2a_0+a_1)=(a_0,a_1)=1$ since $\operatorname{cont}(g)=1$. Thus,

$$\begin{aligned}
&\operatorname{Ext}\left(\mathbb{Z}\oplus\mathbb{Z}[l^{-1}],\ \mathbb{Z}/\left(\sum_{i=0}^2 a_i\right)\mathbb{Z}\right)\\
&\quad=\operatorname{Ext}\left(\mathbb{Z},\ \mathbb{Z}/\left(\sum_{i=0}^2 a_i\right)\mathbb{Z}\right)\oplus\operatorname{Ext}\left(\mathbb{Z}[l^{-1}],\ \mathbb{Z}/\left(\sum_{i=0}^2 a_i\right)\mathbb{Z}\right)=0
\end{aligned}$$

by Proposition 2.2. The result for K_1 follows. □

The case $\deg(g)=3$. Let $g=\sum_{i=0}^{3}a_ix^i\in\mathbb{Z}[x]$ be irreducible, $a\in\mathbb{C}$ a root, and $l=\{a_0, a_3\}$. The six-term exact sequence is:

$$\begin{array}{ccccc}
\mathbb{Z}\oplus\wedge^2\Lambda_g & \xrightarrow{0\oplus(1-\wedge^2M_a)} & \mathbb{Z}\oplus\wedge^2\Lambda_g & \xrightarrow{i_*} & K_0(B_g)\\
{\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0}\\
K_1(B_g) & \xleftarrow{i_*} & \Lambda_g\oplus\mathbb{Z}[l^{-1}] & \xleftarrow{(1-M_a)\oplus(1+M_{a_0a_3^{-1}})} & \Lambda_g\oplus\mathbb{Z}[l^{-1}]
\end{array}$$

Again $a\neq 1$, so $1-M_a:\Lambda_g\to\Lambda_g$ is injective. Thus, $\operatorname{im}\delta_0=\ker M_{(1+a_0a_3^{-1})}$ which is either 0 (if $a_3+a_0\neq 0$) or $\mathbb{Z}[l^{-1}]$ (if $a_3+a_0=0$). Since $\ker\delta_0=\operatorname{im} i_*\simeq\mathbb{Z}\oplus\operatorname{coker}(1-\wedge^2M_a)$, we have $K_0(B_g)$ is either $\mathbb{Z}\oplus\operatorname{coker}(1-\wedge^2M_a)$ (if $a_3+a_0\neq 0$) or an extension of $\mathbb{Z}[l^{-1}]$ by $\mathbb{Z}\oplus\operatorname{coker}(1-\wedge^2M_a)$ (if $a_3+a_0=0$).

The group $K_1(B_g)$ is an extension of $\operatorname{im}\delta_1=\mathbb{Z}\oplus\ker(1-\wedge^2M_a)$ by $\ker\delta_1$ where $\ker\delta_1=\operatorname{im} i_*\simeq\operatorname{coker}(1-M_a)\oplus\operatorname{coker}M_{(1+a_0a_3^{-1})}\simeq\mathbb{Z}/(\sum_{i=0}^{3}a_i)\mathbb{Z}\oplus\operatorname{coker}M_{(1+a_0a_3^{-1})}$. By Lemma 1.5, $\operatorname{coker}M_{(1+a_0a_3^{-1})}\simeq\mathbb{Z}/t\mathbb{Z}$ with $t=(a_3+a_0):l$ if $a_3+a_0\neq 0$. If $a_3+a_0=0$, then $\operatorname{coker}M_{(1+a_0a_3^{-1})}=\mathbb{Z}[l^{-1}]$.

Proposition 2.4. *Let $g(x)=\sum_{i=0}^{3}a_ix^i\in\mathbb{Z}[x]$ be a degree 3 irreducible polynomial and $l=\{a_0, a_3\}$.*

If $a_3+a_0\neq 0$, then $K_0(B_g)=\mathbb{Z}\oplus\wedge^2\Lambda_g/(1-\wedge^2M_a)\wedge^2\Lambda_g$ and $0\to\mathbb{Z}/(\sum_{i=0}^{3}a_i)\mathbb{Z}\oplus\mathbb{Z}/t\mathbb{Z}\to K_1(B_g)\to\mathbb{Z}\oplus\ker(1-\wedge^2M_a)\to 0$ where $t=(a_3+a_0):l$.

If $a_3+a_0=0$, then $0\to\mathbb{Z}\oplus\wedge^2\Lambda_g/(1-\wedge^2M_a)\wedge^2\Lambda_g\to K_0(B_g)\to\mathbb{Z}[l^{-1}]\to 0$ and $0\to\mathbb{Z}/(\sum_{i=0}^{3}a_i)\mathbb{Z}\oplus\mathbb{Z}[l^{-1}]\to K_1(B_g)\to\mathbb{Z}\oplus\ker(1-\wedge^2M_a)\to 0$.

It is straightforward to compute these groups if we impose the restriction that both a_3 and $a_0\in\{1,-1\}$ (so $l=1$). In this case, Λ_g has a basis $e_i=a^i$ ($i=0,1,2$) and is isomorphic to $\mathbb{Z}^3$. Identifying $\wedge^2\mathbb{Z}^3$ with $\mathbb{Z}^3$ ($E_i=e_j\wedge e_k$ with $i,j,k\in\{0,1,2\}$ in cyclic order is a basis of $\wedge^2\mathbb{Z}^3$), the map $\wedge^2M_a$ has matrix form

$$\begin{bmatrix} a_1a_3^{-1} & a_2a_3^{-1} & 1\\ -a_0a_3^{-1} & 0 & 0\\ 0 & -a_0a_3^{-1} & 0\end{bmatrix}.$$

The first, second, and third determinantal divisors of $1-\wedge^2M_a$, i.e., the invariants of the submodule $(1-\wedge^2M_a)\wedge^2\Lambda_g$ in $\wedge^2\Lambda_g$, are 1, 1, and $a_0a_2-a_1a_3^{-1}$ respectively. Thus, $\operatorname{coker}(1-\wedge^2M_a)=\mathbb{Z}/(a_2-a_1)\mathbb{Z}$ if $a_0+a_3\neq 0$, and $\operatorname{coker}(1-\wedge^2M_a)=\mathbb{Z}/(a_2+a_1)\mathbb{Z}$ if $a_0+a_3=0$. We also have, if $a_0+a_3\neq 0$, that

$$\ker(1-\wedge^2M_a)\simeq\begin{cases}\mathbb{Z} & \text{if } a_1=a_2,\\ 0 & \text{otherwise.}\end{cases}$$

If $a_0+a_3=0$, then

$$\ker(1-\wedge^2M_a)\simeq\begin{cases}\mathbb{Z} & \text{if } a_1+a_2=0,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 2.5. *Let $g=\sum_{i=0}^{3}a_ix^i$ be a degree 3 irreducible polynomial in $\mathbb{Z}[x]$ with $|a_0|=|a_3|=1$.*

If $a_0 + a_3 \neq 0$, *then*

$$K_0(B_g) = \mathbb{Z} \oplus \mathbb{Z}/(a_2 - a_1)\mathbb{Z}$$

and

$$K_1(B_g) = \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/(\sum_{i=0}^3 a_i)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a_1 = a_2, \\ \mathbb{Z} \oplus \mathbb{Z}/(\sum_{i=0}^3 a_i)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

If $a_0 + a_3 = 0$, *then*

$$K_0(B_g) = \mathbb{Z}^2 \oplus \mathbb{Z}/(a_1 + a_2)\mathbb{Z}$$

and

$$K_1(B_g) = \begin{cases} \mathbb{Z}^3 \oplus \mathbb{Z}/(\sum_{i=0}^3 a_i)\mathbb{Z} & \text{if } a_1 + a_2 = 0, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/(\sum_{i=0}^3 a_i)\mathbb{Z} & \text{otherwise.} \end{cases}$$

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Proposition 3.1. *Let* $g \in \mathbb{Z}[x]$ *be nonconstant and irreducible. If* τ *is a tracial state on* B_g, *then the range of* τ *on* $K_0(B_g)$ *is* $\mathbb{Z}$.

Proof. Identify Λ_g with $\mathbb{Z}[a, a^{-1}]$ for $a \in \mathbb{C}\backslash\{0\}$ a root of g. If $a = 1$, then $B_g = C(\mathbb{T}^2)$ and $\tau(K_0(B_g)) = \mathbb{Z}$, so assume $a \neq 1$. The map $1 - \alpha_*$ of $K_1(C(\widehat{\Lambda}_g))$ restricts to $1 - M_a$ on $\Lambda_g = H^1(\widehat{\Lambda}_g, \mathbb{Z})$ (by viewing an element of Λ_g as an element of $\widehat{\widehat{\Lambda}}_g$ one obtains a unitary in $C(\widehat{\Lambda}_g))$. Since $a \neq 1$, $1 - M_a$ is injective on Λ_g and thus $\underline{\Delta}_\tau^\alpha(\ker(1 - \alpha_*)) = 0$ where $\underline{\Delta}_\tau^\alpha$ is the group homomorphism from $\ker(1 - \alpha_*)$ to $\mathbb{R}/\tau(K_0(C(\widehat{\Lambda}_g)))$ described in [6]. Thus $\tau(K_0(B_g)) = \tau(K_0(C(\widehat{\Lambda}_g)))$ ([6]). Since $\widehat{\Lambda}_g$ is compact and connected $(\text{cont}(g) = 1)$, $\tau(K_0(C(\widehat{\Lambda}_g))) = \mathbb{Z}$. $\square$

We briefly consider how well the K-groups reflect the $*$- or anti-$*$-isomorphism classes of these algebras. Already, if $\deg(g) = 1$, there are many examples of non-$*$- or anti-$*$-isomorphic algebras with isomorphic K-groups (even if we view $K_0(B_g)$ as an ordered group). By results in [1], it is enough to find $g, h \in \mathbb{Z}[x]$ irreducible of degree one with $|g(1)| = |h(1)| \neq 0$ and $g \neq \pm h$ and $g \neq \pm h^0$. There are also many examples if the degree of the polynomials are two. For example, it is enough to find $g = \sum_{i=0}^2 a_i x^i$ and $h = \sum_{i=0}^2 b_i x^i$ irreducible in $\mathbb{Z}[x]$ with $a_0 \neq a_2$, $b_0 \neq b_2$, $|\sum_{i=0}^2 a_i| = |\sum_{i=0}^2 b_i|$ and $|a_2 - a_0| : \{a_0, a_2\} = |b_2 - b_0| : \{b_0, b_2\}$ but $g \neq \pm h$ and $g \neq \pm h^0$. Choose $a_0, a_1, a_2 \in \mathbb{Z}$ with $a_0 + a_2 \neq 0$, $a_0 + a_1 + a_2 \neq 0$, and both $a_1^2 - 4a_0a_2$ and $(2(a_0 + a_2) + a_1)^2 - 4a_0a_2$ not squares. Letting $b_0 = a_0$, $b_2 = a_2$, and $b_1 = -2(a_0 + a_2) - a_1$ yields one example.

References

1. B. Brenken, *Isomorphism classes of solenoidal algebras* I, Canad. Math. Bull. **36** (1993), 414–418.
2. B. Brenken and P. Jorgensen, *A family of dilation crossed product algebras*, J. Operator Theory **25** (1991), 299–308.
3. G. Elliott, *Are amenable C^*-algebras classifiable*?, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 423–427.
4. B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Theory Dynamical Systems **9** (1989), 691–735.

5. M. Pimsner, *Embedding some transformation group C^*-algebras into AF-algebras*, Ergodic Theory Dynamical Systems **3** (1983), 613–626.

6. ——, *Ranges of traces on K_0 of reduced crossed products by free groups*, Operator Algebras and their Connections with Topology and Ergodic Theory, Lecture Notes in Math., vol. 1132, Springer-Verlag, Heidelberg, 1985, pp. 374–408.

7. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed product C^*-algebras*, J. Operator Theory **4** (1980), 93–118.

8. K. Schmidt, *Algebraic ideas in ergodic theory*, CBMS Regional Conf. Ser. in Math., vol. 76, Amer. Math. Soc., Providence, RI, 1990, p. 94.

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