

UPPER BOUNDS FOR THE DERIVATIVE OF EXPONENTIAL SUMS

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ABSTRACT. The equality

$$\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

is shown, where the supremum is taken for all exponential sums p of the form

$$p(t) = a_0 + \sum_{j=1}^n a_j e^{\lambda_j t}, \quad a_j \in \mathbf{R},$$

with nonnegative exponents λ_j . The inequalities

$$\|p'\|_{[a+\delta, b-\delta]} \leq 4(n+2)^3 \delta^{-1} \|p\|_{[a,b]}$$

and

$$\|p'\|_{[a+\delta, b-\delta]} \leq 4\sqrt{2}(n+2)^3 \delta^{-3/2} \|p\|_{L_2[a,b]}$$

are also proved for all exponential sums of the above form with arbitrary real exponents. These results improve inequalities of Lorentz and Schmidt and partially answer a question of Lorentz.

1. INTRODUCTION AND NOTATION

Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$, $\lambda_j \neq 0$, $j = 1, 2, \dots, n$;

$$E(\Lambda_n) := \left\{ f : f(t) = a_0 + \sum_{j=1}^n a_j e^{\lambda_j t}, a_j \in \mathbf{R} \right\};$$

and

$$E_n := \bigcup_{\Lambda_n} E(\Lambda_n) = \left\{ f : f(t) = a_0 + \sum_{i=1}^n a_i e^{\lambda_i t}, a_i, \lambda_i \in \mathbf{R} \right\}.$$

We will use the norms

$$\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|$$

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and

$$\|f\|_{L_2[a,b]} := \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$$

for functions $f \in C[a, b]$.

Schmidt [3] proved that there is a constant $c(n)$ depending on n so that

$$\|p'\|_{[a+\delta, b-\delta]} \leq c(n)\delta^{-1}\|p\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in (0, (b-a)/2)$. Lorentz [2] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$ there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$, and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n) = cn$.¹ Theorem 2 of this paper shows that Schmidt's inequality holds with $c(n) = 4(n+2)^3$. Our first theorem establishes the sharp inequality

$$|p'(a)| \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}$$

for every $p \in E_n$ with nonnegative exponents λ_j .

2. NEW RESULTS

Theorem 1. *We have*

$$\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

for every $a < b$, where the supremum is taken for all exponential sums $p \in E_n$ with nonnegative exponents. The equality

$$\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}$$

also holds for every $0 < a < b$, where the supremum is taken for all Müntz polynomials of the form

$$p(x) = a_0 + \sum_{j=1}^n a_j x^{\lambda_j}, \quad a_j \in \mathbf{R}, \lambda_j \geq 0.$$

Theorem 2. *The inequalities*

$$\|p'\|_{[a+\delta, b-\delta]} \leq 4(n+2)^3 \delta^{-1} \|p\|_{[a,b]}$$

and

$$\|p'\|_{[a+\delta, b-\delta]} \leq 4\sqrt{2}(n+2)^3 \delta^{-3/2} \|p\|_{L_2[a,b]}$$

hold for every $p \in E_n$ and $\delta \in (0, (b-a)/2)$.

3. PROOFS

To prove Theorem 1 we need some notation. If $\Lambda_n := \{\lambda_1 < \lambda_2 < \dots < \lambda_n\}$ is a set of positive real numbers, then the real span of

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}, \quad x \geq 0,$$

¹ (Added in proof) We can now prove this with $c = 2$; the proof will appear elsewhere.

will be denoted by $M(\Lambda_n)$. It is well known that these are Chebyshev spaces on $[0, \infty)$ (see [1] for instance), so $M(\Lambda_n)$ possesses a unique Chebyshev "polynomial" T_{Λ_n} on $[a, b]$, $0 < a < b$, with the properties

- (i) $T_{\Lambda_n} \in M(\Lambda_n)$,
- (ii) $\|T_{\Lambda_n}\|_{[a, b]} = 1$, and
- (iii) there are $a = x_0 < x_1 < \dots < x_n = b$ so that

$$T_{\Lambda_n}(x_j) = (-1)^{n-j}, \quad j = 0, 1, \dots, n.$$

It is routine to prove (see [1] again) that T_{Λ_n} has exactly n distinct zeros on (a, b) ,

$$(1) \quad \max_{0 \neq p \in M(\Lambda_n)} \frac{|p'(a)|}{\|p\|_{[a, b]}} = \frac{|T'_{\Lambda_n}(a)|}{\|T_{\Lambda_n}\|_{[a, b]}} = |T'_{\Lambda_n}(a)|,$$

and

$$(2) \quad \max_{0 \neq p \in M(\Lambda_n)} \frac{|p(0)|}{\|p\|_{[a, b]}} = \frac{|T_{\Lambda_n}(0)|}{\|T_{\Lambda_n}\|_{[a, b]}} = |T_{\Lambda_n}(0)|.$$

Lemma 3. *Let*

$$\Lambda_n := \{\lambda_1 < \lambda_2 < \dots < \lambda_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_1 < \gamma_2 < \dots < \gamma_n\}$$

so that $0 < \lambda_j \leq \gamma_j$ for each $j = 1, 2, \dots, n$. Then

$$(3) \quad |T'_{\Gamma_n}(a)| \leq |T'_{\Lambda_n}(a)|.$$

Proof. Without loss of generality we may assume that there is an index m , $1 \leq m \leq n$, so that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \neq m$, since repeated applications of the result in this situation give the lemma in the general case. First we show that

$$(4) \quad |T_{\Gamma_n}(0)| < |T_{\Lambda_n}(0)|.$$

Indeed, let $R_{\Gamma_n} \in M(\Gamma_n)$ interpolate T_{Λ_n} at the zeros of T_{Λ_n} and be normalized so that $R_{\Gamma_n}(0) = T_{\Lambda_n}(0)$. Then the Improvement Theorem of Pinkus and Smith [4, Theorem 2] yields

$$|R_{\Gamma_n}(x)| \leq |T_{\Lambda_n}(x)| \leq 1, \quad x \in [a, b].$$

Hence, using (2) with Λ_n replaced by Γ_n , we obtain

$$|T_{\Lambda_n}(0)| = |R_{\Gamma_n}(0)| \leq |T_{\Gamma_n}(0)|,$$

which proves (4). Using the defining properties of T_{Λ_n} and T_{Γ_n} , we can deduce that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 1$ zeros in $[a, b]$ (we count every internal zero without sign change twice). Now assume that (3) does not hold; then

$$|T'_{\Lambda_n}(a)| > |T'_{\Gamma_n}(a)|.$$

This, together with (4), implies that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least one zero in $(0, a)$. Hence $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 2$ zeros in $(0, b]$. This is a contradiction, since

$$T_{\Lambda_n} - T_{\Gamma_n} \in \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}, x^{\gamma_m}\},$$

and every function from the above span can have only at most $n + 1$ zeros in $(0, \infty)$ (see [3]). \square

Proof of Theorem 1. It is sufficient to prove only the second statement of the theorem, the first one can be obtained by the change of variable $x = e^t$. We obtain from (1) and Lemma 3 that

$$\frac{|p'(a)|}{\|p\|_{[a,b]}} \leq \lim_{\delta \rightarrow 0+} \frac{|T'_{\Lambda_n, \delta}(a)|}{\|T_{\Lambda_n, \delta}\|_{[a,b]}} = \lim_{\delta \rightarrow 0+} |T'_{\Lambda_n, \delta}(a)|$$

for every p of the form

$$p(x) = a_0 + \sum_{j=1}^n a_j x^{\lambda_j}, \quad a_j \in \mathbf{R}, \lambda_j > 0,$$

where

$$\Lambda_{n, \delta} := \{\delta, 2\delta, 3\delta, \dots, n\delta\}$$

and $T_{n, \delta}$ is the Chebyshev “polynomial” of $M(\Lambda_{n, \delta})$ on $[a, b]$. From the definition and uniqueness of $T_{\Lambda_{n, \delta}}$ it follows that

$$T_{\Lambda_{n, \delta}}(x) = T_n \left(\frac{2}{b^\delta - a^\delta} x^\delta - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right),$$

where $T_n(y) := \cos(n \arccos y)$. Therefore,

$$\begin{aligned} |T'_{\Lambda_{n, \delta}}(a)| &= |T'_n(-1)| \frac{2}{b^\delta - a^\delta} \delta a^{\delta-1} \\ &= \frac{2n^2}{\delta^{-1}(b^\delta - 1) - \delta^{-1}(a^\delta - 1)} a^{\delta-1} \xrightarrow{\delta \rightarrow 0+} \frac{2n^2}{a(\log b - \log a)} \end{aligned}$$

and the theorem is proved. \square

To prove Theorem 2 we need two lemmas.

Lemma 4. For every set $\Lambda_n := \{\lambda_1 < \lambda_2 < \dots < \lambda_n\}$ of nonzero real numbers there is a point $y \in [-1, 1]$ depending only on Λ_n so that

$$|p'(y)| \leq 2(n+2)^3 \|p\|_{L_2[-1, 1]}$$

for every $p \in E(\Lambda_n)$.

Proof. Take the orthonormal set $\{p_k\}_{k=0}^n$ on $[-1, 1]$ defined by

- (i) $p_k \in \text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_k t}\}$, $k = 0, 1, \dots, n$;
- (ii) $\int_{-1}^1 p_i p_j = \delta_{i,j}$, $0 \leq i \leq j \leq n$.

Writing $p \in E(\Lambda_n)$ as a linear combination of the functions p_k , $k = 0, 1, \dots, n$, and using the Cauchy-Schwartz inequality and the orthonormality of $\{p_k\}_{k=0}^n$ on $[-1, 1]$, we obtain in a standard fashion that

$$\max_{p \in E(\Lambda_n)} \frac{|p'(t_0)|}{\|p\|_{L_2[-1, 1]}} = \left(\sum_{k=0}^n p'_k(t_0)^2 \right)^{1/2}, \quad t_0 \in \mathbf{R}.$$

Let

$$A_k := \{t \in [-1, 1] : |p_k(t)| \geq (n+1)^{1/2}\}, \quad k = 0, 1, \dots, n,$$

and

$$B_k := \{t \in [-1, 1] \setminus A_k : |p'_k(t)| \geq 2(n+2)^{5/2}\}, \quad k = 0, 1, \dots, n.$$

Since $\int_{-1}^1 p_k^2 = 1$, we have

$$m(A_k) \leq (n+1)^{-1}, \quad k = 0, 1, \dots, n.$$

Since $\text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}\}$ is a Chebyshev system, each $\tilde{A}_k := [-1, 1] \setminus A_k$ comprises of at most $k+1$ intervals and each B_k comprises of at most $2(k+1)$ intervals. Therefore,

$$2(n+2)^{5/2} m(B_k) \leq \int_{B_k} |p'_k(t)| dt \leq 4(k+1)\sqrt{n+1},$$

whence

$$\sum_{k=0}^n m(B_k) \leq \frac{2\sqrt{n+1}}{(n+2)^{5/2}} \frac{(n+1)(n+2)}{2} < 1.$$

Now let

$$A := [-1, 1] \setminus \bigcup_{k=0}^n (A_k \cup B_k).$$

Then

$$\begin{aligned} m(A) &\geq 2 - \sum_{k=0}^n m(A_k) - \sum_{k=0}^n m(B_k) \\ &> 2 - (n+1)(n+1)^{-1} - 1 = 0, \end{aligned}$$

so there is a point $y \in A \subset [-1, 1]$ where

$$|p'(y)| \leq 2(n+1)^{5/2}, \quad k = 0, 1, \dots, n.$$

Hence,

$$\left(\sum_{k=0}^n p'_k(y)^2 \right)^{1/2} \leq 2(n+2)^3,$$

and the lemma is proved. \square

Lemma 5. *We have*

$$|p'(0)| \leq 2(n+2)^3 \|p\|_{L_2[-2, 2]} \leq 2(n+2)^3 \|p\|_{[-2, 2]}$$

for every $p \in E_n$.

Proof. Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \dots < \lambda_n\}$ be a fixed set of nonzero real numbers, and let $y \in [-1, 1]$ be chosen by Lemma 4. Let $0 \neq p \in E(\Lambda_n)$. Then

$$q(t) := p(t-y) \in E(\Lambda_n);$$

therefore, applying Lemma 4 to q , we obtain

$$\frac{|p'(0)|}{\|p\|_{L_2[-2, 2]}} \leq \frac{|p'(0)|}{\|p\|_{L_2[-1-y, 1-y]}} = \frac{|q'(y)|}{\|q\|_{L_2[-1, 1]}} \leq 2(n+2)^3,$$

and the lemma is proved. \square

Proof of Theorem 2. Let $t_0 \in [a + \delta, b - \delta]$. Applying Lemma 5 to $q(t) := p(\delta t/2 + t_0)$, we get the theorem. \square

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