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GAUSSIAN ESTIMATES AND HOLOMORPHY OF SEMIGROUPS

EL-MAATI OUHABAZ

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ABSTRACT. We show that if a selfadjoint semigroup T on $L^2(\Omega)$ satisfies a Gaussian estimate $|T(t)f| \le MG(bt)|f|$, $0 \le t \le 1$, $f \in L^2(\Omega)$ (where $G = G(t)_{t\ge 0}$ is the Gaussian semigroup on $L^2(R^N)$ and Ω is an open set of R^N), then T defines a holomorphic semigroup of angle $\frac{\pi}{2}$ on $L^p(\Omega)$, $1 \le p < \infty$. We obtain by duality the same result on $C_0(\Omega)$. Applications to uniformly elliptic operators and Schrödinger operators are given.

0. Introduction

Let Ω be an open set of R^N (with the Lebesgue measure), and consider a selfadjoint semigroup $T = T(t)_{t\ge 0}$ on $L^2(\Omega)$ with generator A. Then T is a holomorphic semigroup with angle $\frac{\pi}{2}$, i.e., T can be extended holomorphically to the maximal domain $\{z, \operatorname{Re} z > 0\}$ (see §1 for the precise definition).

Assume now that T interpolates on $L^p(\Omega)$, $1 \le p < \infty$; that is, there exists for each p, a strongly continuous semigroup T_p on $L^p(\Omega)$ with $T_2 = T$ and satisfying $T_p(t)f = T_2(t)f$ $(t \ge 0)$ for $f \in L^p(\Omega) \cap L^2(\Omega)$. It follows from the Stein interpolation theorem that for $1 < p < \infty$ the semigroup T_p is holomorphic on $L^p(\Omega)$ with angle $\theta_p \ge \frac{\pi}{2}(1 - |\frac{2}{p} - 1|)$ (see Davies [6, p. 23]). However, the case $p = 1$ is more delicate.

In the case where A is an elliptic operator of second order (with some smoothness conditions on its coefficients), Amann [2] showed that T_1 is holomorphic on $L^1(\Omega)$ if Ω is bounded and smooth; his method is based on duality arguments and the result of Stewart [21, 22] on $C_0(\Omega)$. Recently, Arendt and Batty [3] extended the result to an arbitrary open set Ω under Dirichlet boundary conditions. We also note that Kato [12] showed the holomorphy on $L^p(R^N)$, $1 \le p < \infty$, for the Schrödinger operator $A = \Delta - V$ (Δ is the Laplacian and $V = V_+ - V_-$ is a potential).

The purpose of this paper is to extend all these results to more general situations. We show the holomorphy on $L^p(\Omega)$, $1 \le p < \infty$, for elliptic operators under more general boundary conditions without regularity on their coefficients and by assuming minimal regularity on Ω. More precisely, we show in an abstract setting the following result:

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Assume that the semigroup T has a *Gaussian estimate*, i.e.,

$$|T(t)f| \leq MG(bt)|f| \quad \text{for } 0 \leq t \leq 1 \text{ and all } f \in L^2(\Omega) \tag{0.1}$$

where M and b are positive constants and $G = G(t)_{t \geq 0}$ is the Gaussian semigroup on $L^2(R^N)$. Then T_p is holomorphic with angle $\frac{\pi}{2}$ on $L^p(\Omega)$ for $1 \leq p < \infty$.

Such Gaussian estimates hold for uniformly elliptic operators and Schrödinger operators (see [6, Chapter 3; 20, Theorem B.7.1]). Our result is applicable in the following framework.

Assume that A is the operator associated with the following symmetric form

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} D_i u \overline{D_j v}\, dx + \int_\Omega V u \bar{v}\, dx$$

with $a_{ij} = a_{ji} \in L^\infty(\Omega)$ satisfying the ellipticity condition $\sum_{i,j=1}^N a_{ij}(x)\xi_i\bar{\xi}_j \geq c|\xi|^2$ a.e. $x \in \Omega$ and $0 \leq V \in L^1_{\text{loc}}(\Omega)$.

The domain of a is given by $D(a) = W \cap \{u \in L^2(\Omega),\ \int_\Omega V|u|^2 < \infty\}$, where W is a closed subspace of the Sobolev space $H^1(\Omega)$ which contains $H^1_0(\Omega)$.

We obtain the holomorphy of T_p, $1 \leq p < \infty$, in the following cases:

(1) $W = H^1_0(\Omega)$ for Ω any open set of R^N (this corresponds to the Dirichlet boundary conditions).

(2) W satisfies the two following properties:

* $u \in W$ implies $|u| \in W$.
* If $0 \leq u \leq v$, $v \in W$, and $u \in H^1(\Omega)$, then $u \in W$.

In this case we assume that Ω has the extension property (if $W = H^1(\Omega)$, this corresponds to the Neumann boundary conditions).

We recall that in [2], [3], and [12] it is shown that T_1 is holomorphic with some "small" angle. To be precise, it is shown that the estimate

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(L^1(\Omega))} \leq \frac{M}{|\lambda|} \tag{0.2}$$

holds for λ s.t. $\operatorname{Re}\lambda > 0$ (here $M > 0$ is a constant).

In the present paper we show that T_p, $1 \leq p < \infty$, is holomorphic with angle $\frac{\pi}{2}$, that is, the estimate (0.2) holds in $\mathscr{L}(L^p(\Omega))$ in each sector $\Sigma(\theta + \frac{\pi}{2}) := \{\lambda = re^{i\alpha};\ r > 0,\ |\alpha| < \frac{\pi}{2} + \theta\}$, $0 \leq \theta < \frac{\pi}{2}$. This holomorphy in the maximal domain $\{z,\ \operatorname{Re} z > 0\}$ answers positively and in a more general situation a question in Kato's paper [12].

We also study the holomorphy on $C_0(\Omega)$. We show by duality that if the Laplacian on $C_0(\Omega)$ is a generator of a semigroup T_0, then T_0 is holomorphic (Ω is any open set of R^N). This result has been shown by Lumer and Paquet [13, 14]. Our method gives more information on the generator and that T_0 is holomorphic with angle $\frac{\pi}{2}$.

This paper is organized as follows. In §1 we recall some known material on holomorphic semigroups. In §2, we show that Gaussian estimates (0.1) imply the holomorphy on $L^p(\Omega)$, $1 \leq p < \infty$. Finally, §3 is concerned with applications to elliptic operators on $L^p(\Omega)$ and $C_0(\Omega)$.

Remark 0.1. (1) All the semigroups considered in this paper are assumed to be strongly continuous.

(2) If E is a Banach space, we denote by $\mathscr{L}(E)$ the space of bounded linear operators on E and by $||\cdot||_E$ the norm of E.

1. PRELIMINARIES

In this section we recall some known results on holomorphic semigroups. Denote by E a Banach space and by A a generator of a semigroup $T = T(t)_{t\geq 0}$ on E. By $\varrho(A)$ and $\sigma(A)$ we denote respectively the resolvent set and the spectrum of A.

Definition 1.1. (a) The semigroup T is said to be *bounded holomorphic with angle* $\theta \in (\theta, \frac{\pi}{2}]$ if T has an extension to the sector $\Sigma(\theta) := \{z = re^{i\alpha};\ \ r > 0, |\alpha| < \theta\}$ which satisfies the following:

(1) $T(z+z') = T(z)T(z')$, $z, z' \in \Sigma(\theta)$.
(2) $z \to T(z)$ is holomorphic on $\Sigma(\theta)$.
(3) $\lim_{z\to 0,\, z\in\Sigma(\theta)} T(z)f = f$ for each $f \in E$.
(4) For each $\theta' < \theta$ there exists a constant M (depending on θ) s.t. $||T(z)||_{\mathscr{L}(E)} \leq M$ for all $z \in \Sigma(\theta')$.

(b) We say that T is *bounded holomorphic* if there exists $\theta \in (0, \frac{\pi}{2}]$ s.t. T is bounded holomorphic with angle θ.

The following can be found in the books on semigroup theory (see [9, 10, 11, 15, and 17]).

Theorem 1.2. *The semigroup T is bounded holomorphic with angle θ if and only if $\Sigma(\theta + \frac{\pi}{2}) \subset \varrho(A)$, and for each $\theta' < \theta$ $(\theta' > 0)$ there exists a constant M s.t.*

$$||(\lambda - A)^{-1}||_{\mathscr{L}(E)} \leq \frac{M}{|\lambda|} \quad \textit{for all}\ \lambda \in \Sigma\left(\theta + \frac{\pi}{2}\right).$$

Theorem 1.3. *The semigroup T is bounded holomorphic if and only if $\Sigma(\frac{\pi}{2}) \subset \varrho(A)$ and*

$$||(\lambda - A)^{-1}||_{\mathscr{L}(E)} \leq \frac{M}{|\lambda|} \quad \textit{for all}\ \lambda \in \Sigma\left(\frac{\pi}{2}\right).$$

Theorem 1.3 can be deduced from Theorem 1.2 by showing that if $||(\lambda - A)^{-1}||_{\mathscr{L}(E)} \leq \frac{M}{|\lambda|}$ for all $\lambda \in \Sigma(\frac{\pi}{2})$, then there exists some "small" angle $\theta > 0$ such that $\Sigma(\theta + \frac{\pi}{2}) \subset \varrho(A)$, and the same estimate holds in $\Sigma(\theta + \frac{\pi}{2})$.

We give now an interesting situation where the semigroup is holomorphic. Let E be a Hilbert space and denote by $(\cdot, \cdot)$ its scalar product. Assume that A is a selfadjoint generator of a bounded semigroup T on E. Then it follows by the spectral theorem that $(Au, u) \leq 0$ for all $u \in D(A)$. Moreover, we have the following well-known

Proposition 1.4. *The semigroup T is bounded holomorphic with angle $\frac{\pi}{2}$.*

2. L^p AND C_0 HOLOMORPHY

We keep the same notation as in the introduction. Ω is an open set of R^N with the Lebesgue measure and T a selfadjoint semigroup on $L^2(\Omega)$ with generator A. We say that T has a Gaussian estimate for $0 \leq t \leq 1$ if the estimate (0.1) holds for $0 \leq t \leq 1$ and that T has a Gaussian estimate for all $t \geq 0$ if the estimate in (0.1) holds for all $t \geq 0$. We start by the following

Proposition 2.1. *If T has a Gaussian estimate for $0 \leq t \leq 1$, then there exists $w \geq 0$ s.t. the semigroup $e^{-wt}T(t)_{t\geq 0}$ has a Gaussian estimate for all $t \geq 0$.*

Proof. By assumption $|T(t)| \leq MG(bt)$ for $0 \leq t \leq 1$. Let $t \geq 1$, and write $t = n + s$ with $0 \leq s < 1$ and n a natural number. Then

$$\begin{aligned}|T(t)| &= |T(n)T(s)| = |T(1)^n T(s)| \\ &\leq M^{n+1}G(nb)G(sb) = M^{n+1}G(bt) \leq Me^{wt}G(bt)\end{aligned}$$

with $w = \log M$. This shows the proposition. □

The Gaussian semigroup G is contractive on $L^p(R^N)$ for $1 \leq p \leq \infty$. This implies that if (0.1) holds, then the semigroup T satisfies $\|T(t)f\|_{L^p(\Omega)} \leq Me^{wt}\|f\|_{L^p(\Omega)}$ for $f \in L^p(\Omega) \cap L^2(\Omega)$, $1 \leq p \leq \infty$. By the Riesz-Thorin interpolation theorem, there exists $T_p(t) \in \mathscr{L}(L^p(\Omega))$ s.t. $T_p(t)f = T_2(t)f := T(t)f$ for $f \in L^p(\Omega) \cap L^2(\Omega)$, $1 \leq p \leq \infty$. One can see easily that for $1 < p < \infty$, $T_p = T_p(t)_{t\geq 0}$ is a strongly continuous semigroup on $L^p(\Omega)$. Moreover, we have

Proposition 2.2. *T_1 is a strongly continuous on $L^1(\Omega)$.*

By the density argument we see that $T_1(t+s) = T_1(t)T_1(s)$. We do not give the proof of the strong continuity since it is contained in Theorem 2.4.

It is easy to see from (0.1) that $T(t)$ maps $L^2(\Omega)$ into $L^\infty(\Omega)$ for each $t > 0$. This implies in particular that $T(t)$ is given by a kernel $K(t, x, y)$, i.e.,

$$T(t)f(x) = \int_\Omega K(t, x, y)f(x)\,dy \quad \text{for each } t > 0,\ f \in L^2(\Omega),\ \text{and } x \in \Omega.$$

The same is clearly true for $T(z)$ for $z \in \Sigma(\frac{\pi}{2})$. Denote by $K(z, x, y)$ the corresponding kernel of $T(z)$. We have

Proposition 2.3 (see [6, p. 103]). *Assume that T has a Gaussian estimate for all $t \geq 0$. Then there exist two positive constants M and b such that*

$$|K(z, x, y)| \leq \frac{M}{((\operatorname{Re} z)^{N/2}} \exp\left\{-\operatorname{Re}\frac{b(x-y)^2}{z}\right\}$$

for $z \in \Sigma(\frac{\pi}{2})$ and $x, y \in \Omega$.

We state now our result on the holomorphy on $L^p(\Omega)$. With the same notation as above, we have

Theorem 2.4. *If T has a Gaussian estimate for $0 \leq t \leq 1$, then there exists $w \geq 0$ s.t. the semigroup $e^{-wt}T_p(t)_{t\geq 0}$ is bounded holomorphic with angle $\frac{\pi}{2}$ on $L^p(\Omega)$ for $1 \leq p < \infty$.*

Proof. From Proposition 2.1 there exists $w \geq 0$ s.t. the semigroup $e^{-w\cdot}T$ has a Gaussian estimate for all $t \geq 0$. We can then assume that T has a Gaussian estimate for all $t \geq 0$ in order to work with T instead of $e^{-w\cdot}T$.

We let $K_0(z, x, y) = \exp\{-\operatorname{Re}\frac{b(x-y)^2}{z}\}$, $\operatorname{Re} z > 0$, and $k_0(z)$, be the function given by $k_0(z)(x) = \exp\{-\operatorname{Re}\frac{bx^2}{z}\}$, $x \in R^N$.

Let $f \in L^1(\Omega) \cap L^2(\Omega)$. By Proposition 2.3 we have

$$\begin{aligned} |T(z)f(x)| &\leq M((\operatorname{Re} z)^{-N/2} \int_\Omega K_0(z, x, y)|f(y)|\, dy \\ &\leq M(\operatorname{Re} z)^{-N/2} \int_{R^N} K_0(z, x, y)|f^\sim(y)|\, dy \end{aligned}$$

where $f^\sim(y) = f(y)$ if $y \in \Omega$ and 0 if not.

By Young's inequality we get

$$\|T(z)f\|_{L^p(\Omega)} \leq M(\operatorname{Re} z)^{-N/2}\|k_0(z)\|_{L^1(R^N)}\|f\|_{L^p(\Omega)} \quad \text{for } 1 \leq p < \infty \text{ and } \operatorname{Re} z > 0. \tag{2.1}$$

This implies that $T(z)$ can be extended to a bounded operator $T_p(z) \in \mathscr{L}(L^p(\Omega))$, $1 \leq p < \infty$. By density we have $T_p(z + z') = T_p(z)T_p(z')$, $z, z' \in \Sigma(\frac{\pi}{2})$.

Let $\theta \in (0, \frac{\pi}{2})$ be fixed. The inequality (2.1) gives

$$\begin{aligned} \|T_p(z)\|_{\mathscr{L}(L^p(\Omega))} &\leq M((\operatorname{Re} z)^{N/2} \int_{R^N} \exp\left\{-\left(\frac{\sqrt{b}\operatorname{Re} z x}{|z|}\right)^2\right\} dx \\ &\leq M\left(\frac{1}{\sqrt{b}}\right)^N \left(\frac{|z|}{\operatorname{Re} z}\right)^N \int_{R^N} e^{-x^2}\, dx. \end{aligned}$$

It follows that there exists a constant $M_0 > 0$ s.t.

$$\|T_p(z)\|_{\mathscr{L}(L^p(\Omega))} \leq M_0 \left(\frac{1}{\cos\theta}\right)^N \quad \text{for } z \in \Sigma(\theta) \text{ and } 1 \leq p < \infty. \tag{2.2}$$

Consequently, $\|T_p(z)\|_{\mathscr{L}(L^p(\Omega))}$ is bounded in $\Sigma(\theta)$.

We show that $z \to T_p(z)$ is holomorphic from $\Sigma(\theta)$ in $\mathscr{L}(L^p(\Omega))$. It is known that this is equivalent to the weak holomorphy, i.e., $z \to \langle T_p(z)f, g\rangle$ holomorphic for each $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\langle\cdot, \cdot\rangle$ is the pairing between $L^p(\Omega)$ and $L^q(\Omega)$. We treat the important case $p = 1$ (for $p \neq 1$ the proof is exactly the same).

Let $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$. Consider a sequence $(f_n)_n \in L^1(\Omega) \cap L^2(\Omega)$ converging to f in $L^1(\Omega)$. Let $g_n = \chi_{\Omega_n} g$ where Ω_n is a sequence of bounded open sets s.t. $\bigcup_n \Omega_n = \Omega$ and χ_{Ω_n} is the indicator function of Ω_n.

For each $n \geq 0$, $\langle T_1(z)f_n, g_n\rangle = \langle T(z)f_n, g_n\rangle$, so $z \to \langle T_1(z)f_n, g_n\rangle$ is holomorphic since $f_n, g_n \in L^2(\Omega)$. But inequality (2.2) implies that

$$|\langle T_1(z)f_n, g_n\rangle| \leq M_0 \left(\frac{1}{\cos\theta}\right)^N \|f_n\|_{L^1(\Omega)}\|g_n\|_{L^\infty(\Omega)}.$$

Hence, there exists a constant $M_1 > 0$ s.t.

$$|\langle T_1(z)f_n, g_n\rangle| \leq M_1 \left(\frac{1}{\cos\theta}\right)^N \|f\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)} \quad \text{for all } n \geq 0 \text{ and } z \in \Sigma(\theta). \tag{2.3}$$

On the other hand $\langle T_1(z)f_n, g_n\rangle \to \langle T_1(z)f, g\rangle$ when $n \to \infty$. It follows from (2.3) and Vitali's theorem (see [10, Theorem 3.14.1]) that $z \to \langle T_1(z)f, g\rangle$ is holomorphic in $\Sigma(\theta)$.

We show now that $T_1(z)f \to f$ when $z \to 0$, $z \in \Sigma(\theta)$ for each $f \in L^1(\Omega)$. Because of (2.2) it suffices to show this for $f \in L^1(\Omega) \cap L^2(\Omega)$. In this case $T_1(t)f = T(t)f$ for $t \geq 0$ and $T(t)f \to f$ in $L^2(\Omega)$ $(t \downarrow 0)$. Consequently, we can extract a sequence $t_n > 0$ s.t. $T_1(t_n)f(x) \to f(x)$ for a.e. $x \in \Omega$ $(t_n \downarrow 0)$. By the Gaussian estimate and the dominated convergence theorem we have $T_1(t_n)f \to f$ in $L^1(\Omega)$. The holomorphy of $z \to T_1(z)$ implies that $T_1(z)f \to f$ in $L^1(\Omega)$ when $z \to 0$, $z \in \Sigma(\theta)$. We have shown that T_1 is bounded holomorphic with angle θ for all $\theta \in (0, \frac{\pi}{2})$. This implies that T_1 is bounded holomorphic with angle $\frac{\pi}{2}$. The proof is complete.

Our next result can be used in particular for Feller semigroups. Denote by $C_0(\Omega)$ the space of continuous functions in Ω which vanish at infinity. Assume that T has a Gaussian estimate for $0 \leq t \leq 1$ and there exists a semigroup T_0 on $C_0(\Omega)$ s.t. $T_0(t)f = T(t)f$ for $f \in C_0(\Omega) \cap L^2(\Omega)$. We have

Corollary 2.5. *There exists $w \geq 0$ such that the semigroup $e^{-w\cdot}T_0$ is bounded holomorphic with angle $\frac{\pi}{2}$ on $C_0(\Omega)$.*

Proof. As in Theorem 2.4 we can assume that T has a Gaussian estimate for all $t \geq 0$. Denote by A_1 and A_0 the generators of T_1 and T_0, and let $A_\infty = A_1^*$ on $L^\infty(\Omega)$.

We first show that $\Sigma(\pi) \subset \varrho(A_0)$. Since T_1 is holomorphic with angle $\frac{\pi}{2}$, then $\Sigma(\pi) \subset \varrho(A_1) = \varrho(A_\infty)$. We claim that $\varrho(A_\infty) \subset \varrho(A_0)$. In fact, let $\lambda \in \partial\sigma(A_0)$ (the boundary of $\sigma(A_0)$). It is known that this implies that λ is in the approximate spectrum of A_0, i.e., there exists a sequence $f_n \in D(A_0)$ s.t. $\|f_n\|_{C_0(\Omega)} = 1$ and $(\lambda - A_0)f_n \to 0$ in $C_0(\Omega)$ (see, for example, [15, p. 64]). But one can see easily that A_∞ is an extension of A_0 and then $\lambda \in \partial\sigma(A_\infty) = \sigma(A_\infty) \subset (-\infty, 0]$. Hence $\sigma(A_0) = \partial\sigma(A_0) \subset \sigma(A_\infty)$ and the claim is shown.

Now since $(\lambda - A_0)^{-1} = (\lambda - A_\infty)^{-1}$ on $C_0(\Omega)$ for $\lambda > 0$ this equality holds for all $\lambda \in \Sigma(\pi)$ by analytic continuation. Let $\theta \in (0, \frac{\pi}{2})$. Then

$$\begin{aligned} \|(\lambda - A_0)^{-1}\|_{\mathscr{L}(C_0(\Omega))} &\leq \|(\lambda - A_\infty)^{-1}\|_{\mathscr{L}(L^\infty(\Omega))} \\ &= \|(\lambda - A_1)^{-1}\|_{\mathscr{L}(L^1(\Omega))} \leq \frac{M}{|\lambda|} \end{aligned}$$

for all $\lambda \in \Sigma(\theta + \frac{\pi}{2})$ by Theorems 1.2 and 2.4. This shows the corollary.

Remark 2.6. The same conclusion as in Corollary 2.5 holds if we replace $C_0(\Omega)$ by $C(\bar{\Omega})$, the space of continuous functions on $\bar{\Omega}$ with Ω bounded.

3. Applications

3.1. **Elliptic operators.** Let Ω be an open set of R^N as above, and consider on $L^2(\Omega)$ the symmetric form

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} D_i u \overline{D_j v}\, dx + \int_\Omega V u \bar{v}\, dx \quad \text{where } D_i = \frac{\partial}{\partial x_i}.$$

Assume that

$$a_{ij} = a_{ji} \in L^\infty(\Omega) \text{ and real}, \ 1 \leq i, j \leq N. \tag{3.1}$$

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\bar{\xi}_j \geq c|\xi|^2 \quad \text{for a.e. } x \in \Omega \tag{3.2}$$

$$\text{and all } \xi = (\xi_1, \ldots, \xi_N) \in C^N,\ c > 0 \text{ is a constant}.$$

$$0 \leq V \in L^1_{\mathrm{loc}}(\Omega). \tag{3.3}$$

Let W be a closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$ (here $H^1(\Omega)$ and $H^1_0(\Omega)$ are the classical Sobolev spaces).

The domain $D(a)$ of a is given by $D(a) = W \cap \{u \in L^2(\Omega),\ \int_\Omega V|u|^2\,dx < \infty\} = W \cap D(V)$. It is easy to see that the form a with this domain is closed. Denote by T its associated selfadjoint semigroup on $L^2(\Omega)$.

We shall say that W is an ideal of $H^1(\Omega)$ if the following two properties are satisfied:

(1) $u \in W$ implies $|u| \in W$.

(2) If $0 \leq v \leq u$, $u \in W$, and $v \in H^1(\Omega)$, then $v \in W$.

An example of such a W is given by $W = \{u \in H^1(\Omega),\ u_{|\Gamma} = 0\}$ where Γ is a closed set of the boundary of $\partial\Omega$ of Ω. (Here we assume that Ω is bounded and regular.)

Keeping these notation and those of §2, we have

Theorem 3.1. *Under the assumptions* (3.1), (3.2), *and* (3.3) *we have*

(1) *If* $W = H^1_0(\Omega)$ *with* Ω *any open set of* R^N, *then* T_p *is bounded holomorphic with angle* $\frac{\pi}{2}$ *on* $L^p(\Omega)$, $1 \leq p < \infty$.

(2) *If* W *is an ideal of* $H^1(\Omega)$, *we assume in addition that* Ω *has the extension property. Then there exists* $w \geq 0$ *s.t. the semigroup* $e^{-w\cdot}T_p$ *is bounded holomorphic with angle* $\frac{\pi}{2}$ *on* $L^p(\Omega)$, $1 \leq p < \infty$.

Proof. (1) If $W = H^1_0(\Omega)$, then T has a Gaussian estimate for all $t \geq 0$ (see [6, Corollary 3.2.8]). We then apply Theorem 2.4.

(2) If W is an ideal of $H^1(\Omega)$, then the semigroup T corresponding to the form a with domain $D(a) = W \cap D(V)$ is dominated by the semigroup $T_{H^1(\Omega)}$ corresponding to the form a with domain $D(a) = H^1(\Omega) \cap D(V)$ (see [16] or [23]). Since $T_{H^1(\Omega)}$ has a Gaussian estimate for $0 \leq t \leq 1$ (see [6, Theorem 3.2.9]), then so has T. The claim follows again from Theorem 2.4.

Another important situation where we have Gaussian estimates is the case of the Schrödinger operator $\Delta - V$ where $V = V_+ - V_-$ with $0 \leq V_+ \in L^1_{\mathrm{loc}}(R^N)$ and $0 \leq V_-$ is in the Kato class. It is an easy consequence of the Feynman-Kac formula that the semigroup generated by $\Delta - V$ has a Gaussian estimate for $0 \leq t \leq 1$. (See, for example, [1, 20, 7]. In [1], it is shown that Gaussian estimates hold for $\Delta - \mu$ where $\mu = \mu_+ - \mu_-$ is a certain smooth measure.) We then obtain that $\Delta - V$ generates a holomorphic semigroup with angle $\frac{\pi}{2}$ on $L^p(R^N)$ for $1 \leq p < \infty$. The fact that the angle of holomorphy can be $\frac{\pi}{2}$ answers positively a question in [12].

We can also show this result without referring to the Feynman-Kac formula. In fact, it is clear that the semigroup $e^{t(\Delta - V_+)}$ generated by $\Delta - V_+$ is dominated by the Gaussian semigroup. It follows from our result that $e^{t(\Delta - V_+)}$ is bounded holomorphic with angle $\frac{\pi}{2}$ on $L^1(R^N)$. Moreover if V_- is in the Kato class, this means that V_- is Δ-bounded with bound 0 on $L^1(R^N)$. Then V_- is

$\Delta - V_+$-bounded with bound 0. The result follows by the well-known perturbation arguments. This perturbation method has the advantage of working for complex potentials V such that $\operatorname{Im} V$ and $(\operatorname{Re} V)_-$ are in Kato class and $(\operatorname{Re} V)_+ \in L^1_{\mathrm{loc}}(R^N)$. To be precise, the operator $\Delta - V$ is seen here as a perturbation of $\Delta - \operatorname{Re} V$ by $-i \operatorname{Im} V$.

3.2. **Laplacian on** $C_0(\Omega)$. In this subsection we give an application of Corollary 2.5 to the Laplacian Δ on $C_0(\Omega)$ with Ω any open set of R^N. The result has been obtained in [13, 14], but their method does not give that the angle of holomorphy can be $\frac{\pi}{2}$.

Proposition 3.2. *Assume that there exists a realization A_0 of the Laplacian which generates a semigroup on $C_0(\Omega)$. Then $D(A_0) = \{u \in C_0(\Omega),\ \Delta u \in C_0(\Omega)\}$.*

Proof. Let $u \in C_0(\Omega)$ s.t. $\Delta u \in C_0(\Omega)$. Let $\lambda > 0$ $(\lambda \in \varrho(A_0))$. There exists $v \in D(A_0)$ s.t. $\lambda u - \Delta u = \lambda v - A_0 v$. This implies that $\lambda(u-v) - \Delta(u-v) = 0$ (in the distributional sense). It is known by regularity results that this implies $u - v$ is a C^∞ function on Ω (see, e.g., [5]). The maximum principle implies that $u = v$ since $u - v \in C_0(\Omega)$.

Theorem 3.3. *Assume that the Laplacian Δ with domain $D(\Delta) = \{u \in C_0(\Omega),\ \Delta u \in C_0(\Omega)\}$ generates a semigroup T_0 on $C_0(\Omega)$. Then T_0 is bounded holomorphic with angle $\frac{\pi}{2}$.*

Proof. Consider the following form of the Dirichlet Laplacian on $L^2(\Omega)$

$$a(u, v) = \sum_{i=1}^{N} \int_\Omega D_i u \overline{D_i v}\, dx, \qquad D(a) = H_0^1(\Omega).$$

Denote by T_2 its associated semigroup. Then T_2 has a Gaussian estimate for all $t \geq 0$ (more precisely, we have $T_2(t) \leq G(t)$ for $t \geq 0$; see [6, Theorem 2.1.6] or [16, Proposition 4.2]). We have to show that $T_2(t)f = T_0(t)f$ for $f \in C_0(\Omega) \cap L^2(\Omega)$ and apply Corollary 2.5.

If Ω is bounded and sufficiently smooth, then $T_2(t)f = T_0(t)f$ $(f \in C_0(\Omega) \cap L^2(\Omega))$ for all $t \geq 0$ (see, for example, [8, Theorem 8.30; 4, p. 32]).

Let now $(\Omega_n)_{n \geq 1}$ be a sequence of bounded and of class C^∞ open sets s.t. $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\bigcup_n \Omega_n = \Omega$. For each n, define on $L^2(\Omega)$ the form a_n by

$$a_n(u, v) = \sum_{i=1}^{N} \int_{\Omega_n} D_i u \overline{D_i v}\, dx, \qquad D(a) = H_0^1(\Omega_n).$$

Denote by $R(\lambda, a)$ and $R(\lambda, a_n)$ for $\lambda > 0$ the resolvents of the associated operators with a and a_n respectively. The sequence a_n satisfies $a_{n+1} \leq a_n$ in the sense that $D(a_n) \subset D(a_{n+1})$ and $a_{n+1}(u, u) \leq a_n(u, u)$. It follows by a convergence theorem on forms ([11, p. 452] or [18, p. 373]) that $R(\lambda, a_n)$ converges strongly in $L^2(\Omega)$ to $R(\lambda, a)$ for all $\lambda > 0$. (Here the forms a_n are not densely defined, but the convergence theorem is still valid; see [3, §7; 6, p. 62]).

Define now the operators Δ_n on $C_0(\Omega_n)$ by

$$D(\Delta_n) = \{u \in C_0(\Omega_n) \cap H_0^1(\Omega_n),\ \Delta u \in C_0(\Omega_n)\}, \qquad \Delta_n u = \Delta u.$$

Since Ω_n is bounded and regular, $R(\lambda, a_n) = (\lambda - \Delta_n)^{-1}$ on $C_0(\Omega_n) \cap L^2(\Omega_n)$ for all $\lambda > 0$.

Let now $f \in C_0(\Omega)$ with compact support. Then $f \in C_0(\Omega_n)$ for $n \geq n_0$ (for some n_0). Let $h_n = (\lambda - \Delta_n)^{-1} f$ and $h = (\lambda - \Delta)^{-1} f$, $\lambda > 0$. Define $h_n^{\sim}$ by $h_n^{\sim}(x) = h_n(x)$ for $x \in \Omega_n$ and 0 if $x \in \Omega \setminus \Omega_n$. Then $h_n^{\sim} \in C_0(\Omega)$ and $(\lambda - \Delta)(h_n^{\sim} - h) = 0$ on Ω_n with $h_n^{\sim} - h \in C(\bar{\Omega}_n)$. The maximum principle implies that $\|h_n^{\sim} - h\|_{C(\Omega_n)} = \sup_{\partial\Omega_n} |h_n^{\sim} - h| = \sup_{\partial\Omega_n} |h|$. But $h \in C_0(\Omega)$, so $\sup_{\partial\Omega_n} |h|$ converges to 0 when $n \to \infty$. Hence

$$\|(\lambda - \Delta_n)^{-1} f - (\lambda - \Delta)^{-1} f\|_{C(\Omega_n)} \to 0.$$

In particular, $(\lambda - \Delta_n)^{-1} f(x) \to (\lambda - \Delta)^{-1} f(x)$ for all $x \in \Omega$. We then get that $R(\lambda, a) f = (\lambda - \Delta)^{-1} f$. This shows the theorem.

Remark 3.4. The last result is true if instead of the Laplacian we consider an elliptic operator with smooth coefficients.

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SFB 288, Technische Universität Berlin, 136, Strasse 17 Juni, 1000 Berlin 12, Germany

Max-Planck Arbeitsgruppe, FB Mathematik, 10, Am Neuen Palais, 1571 Potsdam, Germany

Current address: Equipe d'Analyse et de Mathématiques Appliqueés, Université de Marne-la-Valleé 2, rue de la Butte Verte, 93166 Noisy-le-Grand, France