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ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT

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ABSTRACT. Let γ denote Euler's constant, and let

$$s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \qquad (n \geq 2).$$

We prove by Ser's formula for the remainder $\gamma - s_n$ that for all integers $n \geq 1$ and $\tau \geq 2$ there are integers $\mu_{n,0}, \mu_{n,1}, \dots, \mu_{n,n}$ such that

$$\mu_{n,0}s_\tau + \mu_{n,1}s_{\tau+1} + \cdots + \mu_{n,n}s_{\tau+n} = \gamma + O_\tau((n(n+1)(n+2) \cdot \cdots \cdot (n+\tau))^{-1}),$$

where the constant in O_τ depends only on τ.

The coefficients $\mu_{n,k}$ are explicitly given and are bounded by $2^{3n+\tau-1}$.

By γ we denote Euler's constant; it is well known that the sequence $(s_n)_{n\geq 0}$ defined by

$$s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \qquad (n \geq 2)$$

tends to γ, where

$$s_n = \gamma + O(n^{-1}) \qquad (n \geq 2).$$

J. Ser [6] has proved that the remainder of $\gamma - s_n$ $(n \geq 2)$ can be expressed as an infinite sum with rational terms: Let

$$t_{m+2} = -\frac{1}{(m+1)!}\int_0^1 (0-x)(1-x) \cdot \cdots \cdot (m-x)\,dx \qquad (m \geq 0). \tag{1}$$

Then

$$\gamma = \frac{1}{n}\sum_{m=0}^{\infty} \frac{t_{m+2}}{\binom{m+n}{m}} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \qquad (n \geq 2). \tag{2}$$

(See also [3, pp. 14–15].)

But, of course, $\gamma - s_n$ can be written in a lot of different ways. For example, we get by *Euler's summation formula* for any positive integers $n \geq 2$ and k:

$$\gamma = s_n + \frac{1}{2n} + \sum_{j=1}^{k} \frac{B_{2j}}{2j \cdot n^{2j}} + R(n, k),$$

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where B_m are the Bernoulli numbers and

$$|R(n,k)| \le \frac{4}{n}\sqrt{\frac{k}{\pi}}\left(\frac{k}{\pi e n}\right)^{2k}$$

(see [4]).

A historical remark. The representation of γ by the right-hand side of (2) was the main tool in P. Appell's attempt to prove the irrationality of γ in 1926 [1]. Appell himself quickly discovered his error and within a week he published a retraction. An outline of this incorrect proof is sketched in [2]. In what follows we apply a linear sequence transformation to the class of those sequences, where the error term can be expressed by a sum like (2). First we introduce some notation:

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdot\cdots\cdot(\alpha+m-1), \quad (\alpha)_0 = 1 \quad (\alpha \in \mathbb{R},\ m \in \mathbb{Z}_{>0});$$

$$\mu_{n,k}(\tau) = \mu_{n,k} = (-1)^{n+k}\frac{(\tau+k)_n}{n!}\binom{n}{k} \qquad (n \in \mathbb{Z}_{\ge 0},\ 0 \le k \le n),$$

where $\tau \in \mathbb{Z}_{>0}$ is fixed. Note that $\mu_{n,k} \in \mathbb{Z}$ $(n \in \mathbb{Z}_{\ge 0},\ 0 \le k \le n)$.

Theorem 1. *Let* $(v_n)_{n\ge 0}$ *be a sequence of real numbers such that*

$$\lim_{n\to\infty} v_n = s,$$

$$v_n = s - \sum_{m=1}^{\infty} \frac{c_m}{(n+\tau)_m} \qquad (n \ge 0), \tag{3}$$

where $(c_m)_{m\ge 1}$ *denotes a sequence of real numbers satisfying*

$$0 \le c_m \le C\cdot(m+\sigma)! \qquad (m \ge \max\{1; -\sigma\}) \tag{4}$$

for some constant $C > 0$ *and some*

$$\sigma \in \mathbb{Z} \quad \textit{with } \sigma < \tau - 2. \tag{5}$$

Then we have for

$$e_n = \left(\sum_{k=0}^{n} \mu_{n,k} v_k\right) - s: \tag{6}$$

$$|e_n| \le C\cdot\frac{(n+\sigma+1)!\cdot(\tau-\sigma-3)!}{(n+\tau-\sigma-2)!} \qquad (n \ge \max\{0; -(\sigma+1)\}).$$

The linear sequence transformation given in (6) belongs to a certain class of so-called nonregular methods; a general theory of such transformations can be found in [7] (see Chapter 2.3.5).

Theorem 2. *For* $n \ge 1$ *and* $\tau \ge 2$ *we have*

$$\left|\sum_{k=0}^{n} \mu_{n,k} s_{k+\tau} - \gamma\right| \le \frac{(\tau-1)!}{2n(n+1)(n+2)\cdot\cdots\cdot(n+\tau)}.$$

From this theorem we get a very good approximation to γ in terms of $s_n, s_{n+1}, \ldots, s_{2n}$ by choosing $\tau = n \ge 2$:

$$\left|\sum_{k=0}^{n} \mu_{n,k} s_{n+k} - \gamma\right| \le \frac{1}{2n^2\binom{2n}{n}} \le n^{-3/2}\cdot 4^{-n}.$$

There are linear sequence transformations for $(s_n)_{n\geq 0}$ with nonintegral coefficients, which converge more rapidly to γ than the transformation given in Theorem 2 (see [5]). But from an arithmetical point of view in number theory it is much more attractive to accelerate the convergence by transformations with integral coefficients.

Proof of the theorems. From $\sum_{k=0}^{n} \mu_{n,k} = 1$ we have by (3) and (6) for every $n \geq 0$:

$$e_n = -\sum_{k=0}^{n}\sum_{m=1}^{\infty}(-1)^{n+k}\frac{(k+\tau)_n}{k!\cdot(n-k)!\cdot(k+\tau)_m}c_m \tag{7}$$

$$= -\sum_{m=1}^{\infty} c_m \sum_{k=0}^{n}(-1)^{n+k}\frac{(k+n+\tau-1)!}{k!\cdot(n-k)!\cdot(k+m+\tau-1)!}. \tag{8}$$

From $c_m \geq 0$ in (4) we conclude that the infinite series $\sum_{m=1}^{\infty}\frac{c_m}{(n+\tau)_m}$ $(n \geq 0)$ converges absolutely, and so we may interchange the sums in (7). We express the terms in (8) again by Pochhammer's symbol; this gives for $n \geq 0$:

$$e_n = (-1)^{n+1}\cdot\frac{(n+\tau-1)!}{n!}\sum_{m=1}^{\infty}\frac{c_m}{(m+\tau-1)!}\sum_{k=0}^{n}\frac{(n+\tau)_k\cdot(-n)_k}{k!\cdot(m+\tau)_k}$$

$$= (-1)^{n+1}\cdot\frac{(n+\tau-1)!}{n!} \cdot\left(\left(\sum_{m=1}^{n}+\sum_{m=n+1}^{\infty}\right)\frac{c_m}{(m+\tau-1)!}\sum_{k=0}^{\infty}\frac{(n+\tau)_k\cdot(-n)_k}{k!\cdot(m+\tau)_k}\right) \tag{9}$$

(since $(-n)_k = 0$ if $k > n$). Let a, b, c be real numbers, $c \neq 0, -1, -2, \dots$;

$$F(a, b; c; x) = \sum_{k=0}^{\infty}\frac{(a)_k\cdot(b)_k}{k!\cdot(c)_k}x^k. \tag{10}$$

We only treat the case $c-a-b > 0$; for this it is well known that

$$F(a, b; c; 1) = \begin{cases} \frac{\Gamma(c)\cdot\Gamma(c-a-b)}{\Gamma(c-a)\cdot\Gamma(c-b)} & \text{if } c-a,\ c-b \neq 0, -1, -2, \dots, \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

The sum on the right-hand side of (10) occurs in (9) with

$$a = n+\tau, \qquad b = -n, \qquad c = m+\tau.$$

From $m \geq 1$ in (9) we have $m+\tau > \tau$, hence $c > a+b$. Note $c-a \leq 0 \Leftrightarrow m \leq n$. By (11) we now see that e_n equals

$$(-1)^{n+1}\frac{(n+\tau-1)!}{n!}\sum_{m=n+1}^{\infty}\frac{c_m}{(m+\tau-1)!}\frac{(m+\tau-1)!\cdot(m-1)!}{(m-n-1)!\cdot(m+n+\tau-1)!}$$

$$= (-1)^{n+1}\frac{(n+\tau-1)!}{n!}\sum_{m=0}^{\infty}c_{m+n+1}\frac{(m+n)!}{m!\cdot(m+2n+\tau)!} \qquad (n \geq 0). \tag{12}$$

Now let

$$n_0 = \max\{0; -(\sigma+1)\}.$$

$n \geq n_0$ implies $m+n+1 \geq n+1 \geq \max\{1, -\sigma\}$. Thus for $n \geq n_0$ we estimate e_n from (12) by (4),

$$(13) \qquad |e_n| \leq C \cdot \frac{(n+\tau-1)!}{n!} \sum_{m=0}^{\infty} \frac{(m+n+\sigma+1)! \cdot (m+n)!}{m! \cdot (m+2n+\tau)!} \qquad (n \geq n_0).$$

We treat the infinite sum in (13) in the same way as we did with the inner sum in (8). For $n \geq n_0$ we get

$$(14) \qquad |e_n| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)!}{(2n+\tau)!} \sum_{m=0}^{\infty} \frac{(n+\sigma+2)_m \cdot (n+1)_m}{m! \cdot (2n+\tau+1)_m}.$$

To apply (11) again we now define in (10):

$$a = n+\sigma+2, \qquad b = n+1, \qquad c = 2n+\tau+1.$$

From (5) we have $2n+\tau+1 > 2n+\sigma+3$, hence $c > a+b$. That gives

$$\begin{aligned} |e_n| &\leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)!}{(2n+\tau)!} \frac{\Gamma(2n+\tau+1) \cdot \Gamma(\tau-\sigma-2)}{\Gamma(n+\tau-\sigma-1) \cdot \Gamma(n+\tau)} \\ &= C \cdot \frac{(n+\sigma+1)! \cdot (\tau-\sigma-3)!}{(n+\tau-\sigma-2)!} \qquad (n \geq n_0). \end{aligned}$$

This proves the theorem.

Theorem 2 follows immediately from Theorem 1 and (2): Put

$$(15) \qquad c_m = -\frac{1}{m} \int_0^1 (0-x)(1-x) \cdot \cdots \cdot (m-1-x)\, dx \qquad (m \geq 1).$$

Hence

$$t_{m+1} = \frac{1}{(m-1)!} \cdot c_m \qquad (m \geq 1);$$

from the definition of s_n and (2) we get[1]

$$\begin{aligned} s_{n+\tau} &= \gamma - \frac{1}{n+\tau} \sum_{m=1}^{\infty} \frac{t_{m+1}}{\binom{m+n+\tau-1}{m-1}} = \gamma - \sum_{m=1}^{\infty} c_m \cdot \frac{(n+\tau-1)!}{(m+n+\tau-1)!} \\ &= \gamma - \sum_{m=1}^{\infty} \frac{c_m}{(n+\tau)_m} \qquad (n \geq 0). \end{aligned}$$

This is (3), where $\tau \geq \mathbb{Z}_{\geq 2}$. We get an integer σ from (15) by

$$0 \leq c_m \leq \frac{1}{m} \int_0^1 x \cdot (m-1)!\, dx = \frac{(m-1)!}{2m} \leq \frac{(m-2)!}{2} \qquad (m \geq 2).$$

Hence we may choose $\sigma = -2$, $C = \frac{1}{2}$, $n_0 = 1$; and (5) holds.

Theorem 2 now follows from Theorem 1, where $v_n = s_{n+\tau}$.

At last note that

$$\begin{aligned} \mu_{n,k} &= (-1)^{n+k} \frac{(\tau+k)_n}{n!} \binom{n}{k} = (-1)^{n+k} \binom{n+k+\tau-1}{n} \binom{n}{k} \\ &= (-1)^{n+k} \binom{n+k+\tau-1}{n-k,\, k,\, k+\tau-1} \qquad (n \geq 1,\ 0 \leq k \leq n) \end{aligned}$$

[1]Note that (2) holds for $s_{n+\tau}$ with $n \geq 0$ and $\tau \geq 2$.

and

$$|\mu_{n,k}| \leq 2^{n+k+\tau-1} \cdot 2^n \leq 2^{3n+\tau-1}.$$

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