PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 5, May 1995

ON HILBERT SPACES WITH UNITAL MULTIPLICATION

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a new simplified proof of two theorems of Froelich, Ingelstam, and Smiley. Our approach enables us also to generalize both of them. In the second section we prove a related theorem which requires different methods for its proof.

0. INTRODUCTION

The study of strictly cyclic operator algebras due to John Froelich pointed out associative Hilbert algebras with identity 1 satisfying $|xy| \leq |x||y|$ and $|1| = 1$ where $|x| = \sqrt{\langle x, x \rangle}$ is a norm derived from the inner product. These algebras were already studied by Ingelstam in [2] who used the analysis of the so called vertex property for Banach algebras. He proved that such algebras are necessarily division algebras.

A simpler proof was given by Smiley in [3] and his proof was in turn greatly simplified by Froelich in his recent paper [1] which is a base point for our investigation. Our paper has three goals:

(i) Froelich used in his proof Gelfand theory and the Riesz representation theorem. As we show even those can be avoided in order to obtain probably the simplest possible proof.

(ii) We shall replace original assumption $|xy| \leq |x||y|$ by a weaker one $|x^2| \leq |x|^2$.

(iii) In some of our results we can avoid the assumption of associativity.

Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{D}$ denote real numbers, complex numbers, quaternions, and octonions, respectively.

1. GENERALIZATIONS OF FROELICH-INGELSTAM-SMILEY THEOREMS

Proposition 1. *Let $\mathscr{A}$ be a real nonassociative pre-Hilbert algebra with identity 1, and suppose that $|a^2| \leq |a|^2$ holds for all $a \in \mathscr{A}$ and $|1| = 1$. Then for every nonzero $a \in \mathscr{A}$ there exists $a^* \in \mathscr{A}$ such that $aa^* = a^*a = 1$.*

Received by the editors June 9, 1993 and, in revised form, August 24, 1993.

1991 *Mathematics Subject Classification.* Primary 46K15; Secondary 46H70.

Key words and phrases. Hilbert space, Hilbert algebra, absolute valued algebra, division algebra, topological zero divisor.

0002-9939/95 $1.00 + $.25 per page

Proof. Suppose that we have $x \in \{1\}^\perp$ with $|x| = 1$. For each $\lambda \in \mathbb{R}$ we have

$$|(\lambda + x)^2|^2 = |\lambda^2 + 2\lambda x + x^2|^2 \le |\lambda + x|^4$$

and so

$$2\lambda^2(1 + \langle 1, x^2\rangle) + 4\lambda\langle x, x^2\rangle + |x^2|^2 - 1 \le 0.$$

This is possible for all real λ only if $1 + \langle 1, x^2\rangle \le 0$. On the other hand

$$|\langle 1, x^2\rangle| \le |1||x^2| \le |x|^2 = 1$$

and so $x^2 = -1$ follows. If $x \in \{1\}^\perp$ is arbitrary, then $x^2 = -|x|^2$ follows. Note that this trivially holds for $x = 0$ as well.

Given a nonzero $a \in \mathscr{A}$ we may decompose $a = \lambda + x$ where $\lambda \in \mathbb{R}$ and $x \in \{1\}^\perp$. Since $a \neq 0$, we have $\lambda^2 + |x|^2 = |a|^2 \neq 0$ and so we may define $a^* = \frac{1}{\lambda^2 + |x|^2}(\lambda - x)$. Using the above paragraph, we can easily compute $aa^* = a^*a = 1$.

If we use Proposition 1 and the well-known fact that every associative division normed algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we obtain

Corollary 1 (the first Froelich-Ingelstam-Smiley theorem). *Let $\mathscr{A}$ be a real associative pre-Hilbert algebra with identity* 1, *and suppose that $|ab| \le |a||b|$ holds for all $a, b \in \mathscr{A}$ and $|1| = 1$. Then $\mathscr{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

However if we base our proof on the concept of the absolute valued algebra rather than on division normed algebras, then the closer inspection of the proof of Proposition 1 gives us the following generalization of Corollary 1:

Theorem 1. *Let $\mathscr{A}$ be alternative real pre-Hilbert algebra with identity* 1. *Suppose that $|a^2| \le |a|^2$ holds for all $a \in \mathscr{A}$ and $|1| = 1$. Then $\mathscr{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$.*

Proof. Let us recall first that algebra is called alternative if $a^2b = a(ab)$ and $ba^2 = (ba)a$ for all $a, b \in \mathscr{A}$. Every associative algebra is obviously alternative while $\mathbb{D}$ is alternative but not associative.

Next we recall from the proof of Proposition 1 that for each $x \in \{1\}^\perp$ the equality $x^2 = -|x|^2$ holds. This implies that $|a^2| = |a|^2$ in fact holds for all $a \in \mathscr{A}$ since, if we decompose $a = \lambda + x$,

$$|a^2| = |\lambda^2 + 2\lambda x - |x|^2| = \sqrt{(\lambda^2 - |x|^2)^2 + 4\lambda^2|x|^2} = \lambda^2 + |x|^2 = |a|^2.$$

In our first step we shall assume that $1, x, y$ are pairwise orthogonal. Then

$$(x + y)^2 = -|x + y|^2 = -|x|^2 - |y|^2 = x^2 + y^2$$

implies $xy = -yx$. This further implies, together with the Moufang identity $xy \cdot yx = x \cdot y^2 \cdot x$ which is valid in every alternative algebra,

$$|xy|^2 = |(xy)^2| = |xy \cdot xy| = |xy \cdot yx| = |xy^2x| = |x|^2|y|^2.$$

In our second step we shall take x, y both orthogonal to 1. In the same way as in the above paragraph we can verify $xy + yx = -2\langle x, y\rangle$. Decompose $xy = \langle 1, xy\rangle + z$ and $yx = \langle 1, yx\rangle + z_1$. Since $xy + yx \in \mathbb{R}\ 1$ and $z + z_1 \in \{1\}^\perp$, we have $z_1 = -z$. From $x(xy) = x^2y = -|x|^2y$ we obtain $\langle 1, xy\rangle x + xz = -|x|^2y$. From $(yx)x = yx^2 = -|x|^2y$ we obtain

$$\langle 1, yx\rangle x - zx = -|x|^2y = \langle 1, xy\rangle x + xz.$$

But $xz+zx=-2\langle x, z\rangle \in \mathbb{R}\ 1$ while $x\in\{1\}^\perp$, so we have $\langle 1, xy\rangle = \langle 1, yx\rangle$ and $\langle z, x\rangle = 0$. Therefore

$$\langle 1, xy\rangle = \langle 1, yx\rangle = -\langle x, y\rangle, \tag{1}$$

$$\langle xy, x\rangle = \langle yx, x\rangle = 0 \tag{2}$$

if $x, y\in\{1\}^\perp$. Now we shall prove that $|xy| = |x||y|$. If $x=0$, then the result is trivial. Otherwise define

$$y_1 = \frac{-\langle x, y\rangle}{|x|^2}x + y$$

so that x is orthogonal to y_1. According to the above paragraph, we have $|xy_1| = |x||y_1|$. Thus

$$|\langle x, y\rangle + xy|^2 = |x|^2(|y|^2 - \frac{\langle x, y\rangle^2}{|x|^2}) = |x|^2|y|^2 - \langle x, y\rangle^2.$$

According to (1), we have

$$|\langle x, y\rangle + xy|^2 = \langle x, y\rangle^2 + 2\langle x, y\rangle\langle 1, xy\rangle + |xy|^2 = |xy|^2 - \langle x, y\rangle^2$$

and finally $|xy| = |x||y|$.

In our last step we take any $a, b\in\mathscr{A}$ and decompose $a=\lambda+x$, $b=\mu+y$. Then

$$\begin{aligned}|a|^2|b|^2 - |ab|^2 = \lambda^2\mu^2 &+ \lambda^2|y|^2 + \mu^2|x|^2 \\ &+ |x|^2|y|^2 - \lambda^2\mu^2 - \lambda^2|y|^2 - \mu^2|x|^2 - |xy|^2 \\ &- 2\lambda\mu(\langle 1, xy\rangle + \langle x, y\rangle) - 2\lambda\langle y, xy\rangle - 2\mu\langle x, xy\rangle,\end{aligned}$$

so, by (1) and (2), it follows that $|ab| = |a||b|$. Thus $\mathscr{A}$ is an absolute valued algebra with identity and consequently isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{D}$.

Theorem 2 (generalization of the second Froelich-Ingelstam-Smiley theorem). *Let $\mathscr{A}$ be a nonassociative complex pre-Hilbert algebra with identity* 1, *and suppose that $|a^2|\le|a|^2$ holds for all $a\in\mathscr{A}$ and $|1|=1$. Then $\mathscr{A}$ is isomorphic to $\mathbb{C}$ and is consequently automatically associative.*

Proof. If $\mathscr{A}$ were not isomorphic to $\mathbb{C}$, then it would be at least two-dimensional over $\mathbb{C}$ and so there would exist some $x\in\{1\}^\perp$ with $|x|=1$. If we define a real inner product on $\mathscr{A}$ by $\langle a, b\rangle_1 = \operatorname{Re}\langle a, b\rangle$, then $\mathscr{A}$ with this new inner product satisfies the assumptions of Proposition 1. Moreover x and ix are both orthogonal to 1 and so $x^2=(ix)^2=-1$ should hold which is clearly impossible.

We shall finish this section with an example which throws some light on the nonassociative case.

Example 1. Let $\mathscr{H}$ be a Hilbert space with dimension greater than one, and define the multiplication in $\mathscr{A}=\mathbb{R}\oplus\mathscr{H}$ by

$$(\alpha\oplus x)(\beta\oplus y) = (\alpha\beta - \langle x, y\rangle)\oplus(\alpha y+\beta x)$$

and the inner product by

$$\langle(\alpha\oplus x), (\beta\oplus y)\rangle = \alpha\beta + \langle x, y\rangle.$$

Then $\mathscr{A}$ satisfies $|ab| \leq |a||b|$ and $|1| = 1$. However $\mathscr{A}$ contains divisors of zero, and therefore the existence of a^* which satisfies $aa^* = a^*a = 1$ (see Proposition 1) is not a sufficiently restrictive condition in the general nonassociative case. We do not see an easy way to describe all nonassociative algebras satisfying the assumptions of Proposition 1. Note that Example 1 is well known in the theory of Jordan algebras.

2. Algebras satisfying $|x^2| = |x|^2$

It is obvious that we cannot drop the existence of an identity element in the Froelich-Ingelstam-Smiley theorems. We can in fact produce a very trivial example. If $\mathscr{A}$ is any pre-Hilbert space and we define $ab = 0$ for all $a, b \in \mathscr{A}$, then $\mathscr{A}$ is associative, $|ab| \leq |a||b|$, but $\mathscr{A}$ is not isomorphic to one of the algebras from these theorems. It is the purpose of this section to prove that if we change the inequality $|x^2| \leq |x|^2$ to the strict equality, then the existence of identity can be dropped.

Theorem 3. *Let $\mathscr{A}$ be a real associative pre-Hilbert algebra satisfying $|a^2| = |a|^2$ for all $a \in \mathscr{A}$. Then $\mathscr{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. First we shall assume that $\mathscr{A}$ is commutative. Then

$$|\langle a, b\rangle| \leq |ab| \leq |a||b| \tag{3}$$

holds for all $a, b \in \mathscr{A}$ and consequently $\mathscr{A}$ is a normed algebra.

In fact

$$\begin{aligned}|a+b|^2 &= |(a+b)^2| = |a^2 + b^2 + 2ab| \\ &\leq |a^2| + |b^2| + |2ab| = |a|^2 + |b|^2 + 2|ab|\end{aligned}$$

implies $\langle a, b\rangle \leq |ab|$. If we replace a by $-a$, we get $|\langle a, b\rangle| \leq |ab|$.

Now assume for a moment that $|a| = |b| = 1$. Then

$$\begin{aligned}4|ab| &= |(a+b)^2 - (a-b)^2| \\ &\leq |(a+b)^2| + |(a-b)^2| = |a+b|^2 + |a-b|^2 = 4\end{aligned}$$

and so $|ab| \leq 1$. In the general case we can reason as follows:

If $a = 0$ or $b = 0$, then $|ab| \leq |a||b|$ is obvious. Otherwise $|\frac{a}{|a|} \cdot \frac{b}{|b|}| \leq 1$ and so (3) follows.

Now we shall use the well-known fact that a commutative associative real normed algebra without topological zero divisors is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Our next goal is therefore to prove that the algebra under consideration does not have any topological zero divisors.

Suppose that $|a| = 1$, $|x_n| = 1$, and $ax_n \to 0$. By (3) we have

$$|\langle a, x_n\rangle| \leq |ax_n| \to 0.$$

Since $\mathscr{A}$ is associative,

$$|\langle a^2, x_n^2\rangle| \leq |a^2 x_n^2| = |(ax_n)^2| = |ax_n|^2 \to 0.$$

If we compute $|a + x_n|$ in a direct way, we obtain

$$|a + x_n|^4 = (2 + 2\langle a, x_n\rangle)^2 \to 4.$$

If we use the square multiplicativity of the norm, we obtain

$$\begin{aligned}|a+x_n|^4 &= |(a+x_n)^2|^2 = |a^2+2ax_n+x_n^2|^2 \\ &= |a^2|^2+4|ax_n|^2+|x_n^2|^2+4\langle a^2\,,\,ax_n\rangle+4\langle ax_n\,,\,x_n^2\rangle+2\langle a^2\,,\,x_n^2\rangle.\end{aligned}$$

Since

$$|a^2|^2=|a|^4=1\,,\qquad |x_n^2|^2=1\,,$$
$$|\langle a^2\,,\,ax_n\rangle|\le|a^2||ax_n|=|ax_n|\to 0\,,$$
$$|\langle ax_n\,,\,x_n^2\rangle|\le|ax_n||x_n^2|=|ax_n|\to 0\,,$$

we have (note that $\langle a^2\,,\,x_n^2\rangle\to 0$ was already established) that $|a+x_n|^4\to 2$ which contradicts the previously obtained fact.

Now that we proved the result for the commutative case, we can handle the noncommutative one by means of localization. Take some nonzero $b\in\mathscr{A}$. A subalgebra $\mathrm{Gen}(b)$, generated by b, is commutative and so it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Note that it is trivial that this subalgebra also satisfies the assumptions of our theorem. In particular this subalgebra contains the identity element which we denote by e. Then e is of course an idempotent of $\mathscr{A}$. According to Theorem 1 it remains to prove that e is the identity of $\mathscr{A}$.

Given an arbitrary $a\in\mathscr{A}$ we have $e(a-ea)=0$ and $(a-ae)e=0$. Since $e\neq 0$, it remains to prove that $\mathscr{A}$ cannot contain any zero divisors. If $xy=0$ with $|x|=|y|=1$, then $|yx|^2=|(yx)^2|=|yxyx|=0$ and so $yx=0$. Thus

$$|x+y|^2=|(x+y)^2|=|x^2+y^2|=|x-y|^2$$

implies $|x+y|^2=2$. Next we have

$$\begin{aligned}4=|x+y|^4&=|x^2+y^2|^2=|x^2|^2+|y^2|^2+2\langle x^2\,,\,y^2\rangle\\ &=|x|^4+|y|^4+2\langle x^2\,,\,y^2\rangle=2+2\langle x^2\,,\,y^2\rangle\end{aligned}$$

and so $\langle x^2\,,\,y^2\rangle=1$ implies $x^2=y^2$. But then

$$1=|x|^4=|x^2|^2=|x^4|=|x^2y^2|=0$$

is a contradiction.

Acknowledgment

The author wishes to express his gratitude to Professor Angel Rodriguez for his helpful comments.

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