

POSITIVE UNSTABLE PERIODIC SOLUTIONS FOR SUPERLINEAR PARABOLIC EQUATIONS

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ABSTRACT. In this paper, we are concerned with a superlinear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = u^p + h(t, x), & (t, x) \in \mathbf{R}_+ \times \Omega, \\ u = 0, & (t, x) \in \mathbf{R}_+ \times \partial\Omega, \\ u > 0, & (t, x) \in \mathbf{R}_+ \times \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, h is T -periodic with respect to the first variable, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ if $N \leq 2$. It is shown that there exist a stable and an unstable positive T -periodic solution for this problem if h is sufficiently small in L^∞ .

1. INTRODUCTION

Let Ω be a bounded domain in \mathbf{R}^N with smooth boundary $\partial\Omega$ and $h: \mathbf{R}_+ \times \overline{\Omega} \rightarrow \mathbf{R}$ be a bounded function which is T -periodic with respect to the first variable. In this paper, we are concerned with stable and unstable T -periodic solutions for a semilinear parabolic equation

$$(P) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = u^p + h(t, x) & \text{in } \mathbf{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbf{R}_+ \times \partial\Omega, \\ u(t) = u(t+T) & \text{in } \overline{\Omega}_1, \quad u > 0 \text{ in } \mathbf{R}_+ \times \Omega, \end{cases}$$

where $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < +\infty$ if $N \leq 2$.

In the recent decade, many authors have studied the existence and multiplicity of periodic solutions for semilinear parabolic boundary value problems of the form

$$(*) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = g(t, x, u) & \text{in } \mathbf{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbf{R}_+ \times \partial\Omega, \end{cases}$$

where $g: \mathbf{R}_+ \times \Omega \times \mathbf{R} \times \mathbf{R}$ is T -periodic with respect to the first variable. One can find extensive references in [2] for the existence of periodic solutions of $(*)$.

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On the other hand, the stability of periodic solutions of a semilinear parabolic boundary value problem has been studied by Alikakos, Hess, and Matano [1], Hess [12], Hirsch [15], and the authors. The method employed in these works in the sub- and supersolution method. If there exist a subsolution \underline{u} and a supersolution \bar{u} of the problem (*) satisfying $\underline{u} < \bar{u}$, one can find a stable periodic solution of the problem (cf. Dancer and Hess [6]). A few sufficient conditions for the existence of sub- and supersolutions for (*) is known in case that g has a sublinear growth (cf. Hess [12]). The existence of unstable solutions for (*) is considered in Hirano [13] in the case that g has an asymptotically linear nonlinearity.

On the other hand, it seems that only a few results are known when the nonlinearity is superlinear. Esteban [7] obtained the existence of a positive periodic solution of the problem (*) in the case of $g(t, x, \xi) = m(t)\xi^p$ with $1 < p < (3N + 8)/(3N - 4)$. She improved her results in [8] to the case that $1 < p < N/(N - 2)$. It is proved in Ôtani [16] that there exists a periodic solution for (P) if h is sufficiently small with $1 < p < (N + 2)/(N - 2)$.

In this paper, we show that the problem (P) possesses at least one stable and one unstable solution for small h . To show the existence of solutions of (P), we calculate the Leray-Schauder degree of an associated compact mapping by using the method employed in Esteban [7]. The stability and instability of solutions of (P) are investigated in §3.

2. EXISTENCE OF PERIODIC SOLUTIONS FOR (P)

In this section, we first describe our theorem and then evaluate the Leray-Schauder degree to show the existence of positive periodic solutions. Throughout the rest of this paper, we fix $q > N + 1$. For simplicity, put $Q_T = (0, T) \times \Omega$. Let

$$W^{1,2,r}(Q_T) = \{u: Q_T \rightarrow \mathbf{R} \mid u_{x_i x_j}, u_{x_i}, u_t \in L^r(Q_T) \text{ for } 1 \leq i, j \leq N\}$$

and

$$C^{1,2}(\bar{Q}_T) = \{u: \bar{Q}_T \rightarrow \mathbf{R} \mid U_{x_i x_j}, u_{x_i}, u_t \in C(\bar{Q}_T) \text{ for } 1 \leq i, j \leq N\},$$

where $u_{x_i x_j} = \partial^2 u / \partial x_i \partial x_j$, $u_{x_i} = \partial u / \partial x_i$, and $u_t = \partial u / \partial t$. We set $E = W^{1,2,q}(Q_T)$ and denote by $\|\cdot\|$, $\|\cdot\|_q$, and $\langle\langle \cdot, \cdot \rangle\rangle$ the norms of E , $L^q(Q_T)$, and the inner-product of $L^2(Q_T)$, respectively. $B_r(v)$ stands for the closed ball in E with radius r centered at v .

A solution u of (P) is said to be stable if for given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $v_0 \in L^q(\Omega)$ with $|v_0 - u(0)|_q < \delta$, there exists the global solution v for the initial value problem

$$(I) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = v^p + h(t, x) & \text{in } (0, \infty) \times \Omega, \\ v = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

and it holds that $|v(t) - u(t)|_q < \varepsilon$ for all $t > 0$. Here $|\cdot|_q$ stands for the norm of $L^q(\Omega)$. A T -periodic solution u is called unstable if u is not stable. The purpose of this paper is to prove the following result.

Theorem. Let $h \in L^\infty(\mathbf{R}_+ \times \Omega)$ be nonnegative and T -periodic with respect to the first variable. If $\|h\|_{L^\infty}$ is sufficiently small, then there exist at least one stable and one unstable solution of (P) in E .

We denote by λ_1 and φ_1 the first eigenvalue of $-\Delta$ in Ω with the Dirichlet boundary respectively. Let $L = \partial/\partial t - \Delta$ with domain

$$D(L) = \{u \in E : u(T) = u(0) \text{ in } \Omega\}.$$

A compact operator $K: E \rightarrow E$ can be defined by L^{-1} since there is uniquely $u \in E$ satisfying $Lu = v$ for any $v \in E$ arbitrarily given. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the extension of ξ^p defined by $f(\xi) = 0$ for all $\xi < 0$, and let $g(t, x, \xi) = f(\xi) + h(t, x)$. The solutions for (P) are fixed points of the operator $K \circ g$ from E into itself. We first investigate the Leray-Schauder degree of $I - K \circ g$ near 0 and then in a large ball in E . We denote by $\deg(F, U, v)$ the degree of compact operator F on an open set U with respect to v .

Lemma 1. For given $c > 0$, there exist positive numbers $r < c$ and δ such that for each $h \in L^\infty(Q_T)$ with $\|h\|_\infty \leq \delta$,

$$\deg(I - K \circ g, B_r(0), 0) = 1.$$

Proof. According to the standard regularity theory, there exists $C > 0$ such that

$$\|u\| \leq C \|g(t, x, u)\|_q$$

for any solution $u \in D(L)$ for $Lu = sg(t, x, u)$ with $s \in [0, 1]$. Let $c > 0$. Take $0 < \rho < 1$ satisfying $\rho/C_1 < c$ and $\xi^{p+1} \leq \frac{\lambda_1}{2} \xi^2$ for all $\xi \in \mathbf{R}$ with $|\xi| \leq \rho$, where C_1 is the constant of the embedding of E into $C(\overline{Q}_T)$. We put $r = \rho/C_1$. Let $\delta > 0$ such that

$$\frac{\lambda_1}{2} \left(\frac{1}{C} r - \delta \right)^q - \delta r > 0$$

and $\|h\|_\infty \leq \delta$. Suppose that there is a solution $u \in D(L) \cap \partial B_r(0)$ for $Lu = sg(t, x, u)$ for some $s \in [0, 1]$. Then noting that $p q > 2$, we have

$$\begin{aligned} 0 &= \langle u_t - \Delta u - sg(t, x, u), u \rangle \\ &= \int_0^T \int_\Omega (|\nabla u|^2 - su^{p+1} - shu) dx dt \\ &\geq \frac{\lambda_1}{2} \int_0^T \int_\Omega u^{pq} dx dt - \int_0^T \int_\Omega hu dx dt \\ &\geq \frac{\lambda_1}{C} \left(\frac{1}{C} \|u\| - \|h\|_\infty \right)^q - \|h\|_\infty \|u\| > 0. \end{aligned}$$

This contradiction implies the nonexistence of solutions of $Lu = sg(t, x, u)$ on $\partial B_r(0)$. From the homotopy invariance of the Leray-Schauder degree, it follows that

$$\deg(I - K \circ g, B_r(0), 0) = \deg(I, B_r(0), 0) = 1.$$

The following lemma is proved by Esteban for $1 < p < (N + 2)/N$ in [7] and for $1 < p < N/(N - 2)$ in [8] in the case that the right-hand side of (P) is $m(t, x)u^p$. The proof of the following lemma is almost exactly as in [7] and [8] (cf. [10]).

Lemma 2. For each $h \in L^\infty(Q_T)$, there exists $R > r$ such that

$$\deg(I - K \circ g, B_R(0), 0) = 0.$$

Proof. The proof of this lemma based on the same idea as in the proof of Theorem 4 and the alternative proof of Proposition 12 of [7]. Then we will only sketch the proof. We first get the uniform estimate for solutions of (P) in $L^\infty(Q_T)$. Indeed, on the contrary, assume that there is a sequence $\{u_n\}$ of solutions of (P) such that $M_n = \sup_{Q_T} u_n$ diverges to $+\infty$. Let $(t_n, P_n) \in Q_T$ with $M_n = u_n(t_n, P_n)$ and $\{(t_n, P_n)\}$ converge to some (\bar{t}, P) . In case of $(\bar{t}, P) \in Q_T$, put

$$\lambda_n^{2/(p-1)} M_n = 1, \quad y = \frac{x - P_n}{\lambda_n}, \quad \tau = \frac{t - t_n}{\lambda_n^2}, \quad v_n(\tau, y) = \lambda_n^{2/(p-1)} u_n(t, x).$$

By the same argument as in [7], $\{v_n\}$ converges to some $v \in C^{1,2}(\mathbf{R} \times \mathbf{R}^N)$ in $C^{1,2}([-R, R] \times \overline{B_R(0)})$ for any $R > 0$. Then v satisfies $v_t - \Delta v = cv^p$ in $\mathbf{R} \times \mathbf{R}^N$ and $v(0, 0) = 1$. On the other hand, multiplying (P) with u replaced by u_n by $(u_n)_t$ and integrating on Q_T , it follows that

$$\int_0^T \int_\Omega (u_n)_t^2 dx dt \leq \int_\omega \int_\Omega h^2 dx dt.$$

Therefore, for any $R > 0$ and sufficiently large n , we have

$$\int_{-R}^R \int_{B_R(0)} (v_n)_\tau^2 dy d\tau \leq \lambda_n^{4/(p-1)-N+2} \int_0^T \int_\Omega h^2 dx dt \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence $v_t \equiv 0$ in $\mathbf{R} \times \mathbf{R}^N$. According to a Liouville theorem in [9], v is identically zero, which contradicts $v(0, 0) = 1$. In case that $(\bar{t}, P) \in [0, T] \times \partial\Omega \cup \{0, T\} \times \Omega$, we obtain a contradiction by taking the same process as in [7] using that $v_t \equiv 0$. The above argument shows that there exists $L_0 > 0$ such that any solution u of (P) satisfies $\|u\|_{L^\infty} < L_0$. We can now complete the proof by the same argument as in the proof of Theorem 4 of [7]. That is, we define a family of compact operators $\{T_s\}$ from E into itself as follows: $T_s u = v$ if and only if v is a solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = v^p + su + s + h(t, x), & (t, x) \in \mathbf{R}_+ \times \Omega, \\ v = 0, & (t, x) \in \mathbf{R}_+ \times \partial\Omega, \\ v(t) = v(t + T) & \text{in } \overline{\Omega}, v > 0 \text{ in } \mathbf{R}_+ \times \Omega. \end{cases}$$

From the L^∞ estimate above, we have that there exists $C(s) > 0$ such that if u is a fixed point of T_s , then $\|u\| \leq C(s)$. On the other hand, we can see that T_s has no fixed point if s is sufficiently large. Then we have that there exists $R > 0$ such that $\|u\| \leq R$ if u is a fixed point of T_s for some $s > 0$. Then since $\deg(I - T_s, B_R(0), 0) = 0$ for s sufficiently large, the assertion follows from the homotopy invariance of degree.

3. STABILITY AND INSTABILITY OF PERIODIC SOLUTIONS

We discuss the stability and the instability for solutions of (P) in the present section. Let u be a T -periodic solution for (P). Denote by $S(t, s)$ the evolution

operator for the following problem:

$$(LI) \quad \begin{cases} \frac{dv}{dt} - \Delta v = f'(u)v & \text{in } (s, \infty) \times \Omega, \\ v = 0 & \text{on } (s, \infty) \times \partial\Omega, \\ v(s) = z & \text{in } \Omega; \end{cases}$$

that is, $S(t, s)z = v(t)$. Then nonzero eigenvalues of the periodic map $U(t) = S(t + T, t)$ is independent of t (see [11]). It is known that if $U(t)$ satisfies

$$\sigma(U(t)) \cap \{\mu: |\mu| > 1\} \neq \emptyset,$$

where $\sigma(A)$ means the set of eigenvalues of a linear operator A , then u is unstable (see [11, Theorem 8.1.2]). By the result due to Beltramo and Hess [3], we have that the linear operator $L_u = L - f'(u)$ with domain $D(L)$ has a unique real principal eigenvalue μ_u with an associated positive eigenfunction φ_u . It is known that u is stable if $\mu_u > 0$ and u is unstable if $\mu_u < 0$ (cf. [3, 11]). We use a similar idea employed in [13] to prove the following result.

Lemma 3. *Let u be a solution of (P). Then there exists $\rho > 0$ satisfying the following conditions:*

- (i) *If $\mu_u > 0$, then $\deg(I - K \circ g, B_\rho(u), 0) = 1$.*
- (ii) *If $\mu_u = 0$, then $\deg(I - K \circ g, B_\rho(u), 0) = 0$.*

Proof. Suppose that $\mu_u > 0$. Then $K \circ f'(u)$ has no real eigenvalue larger than or equal to 1 (cf. [3]). Then we have $\deg(I - K \circ f'(u), B_r(0), 0) = 1$ for any $r > 0$ (cf. [5, Theorem 8.10]). From the definition of μ_u , we have that

$$\langle Lv - f'(u)v, v \rangle \geq \mu_u \|v\|_2^2$$

for all $v \in E$. Then it is easy to see that there exists $r > 0$ such that

$$\langle Lv - g(t, x, v), v \rangle > 0 \quad \text{for all } v \in B_r(u) \setminus \{u\}.$$

This implies that u is an isolated solution of (P) and by the homotopy invariance of degree, we find that

$$\deg(I - K \circ g, B_\rho(0), 0) = \deg(I - K \circ f'(u), B_\rho(0), 0) = 1$$

for some $\rho > 0$ sufficiently small. We next prove statement (ii). Define \tilde{L} , \tilde{K} , and \tilde{f} by $\tilde{L}(u) = Lu + u$ for $u \in D(L)$, $\tilde{K} = (\tilde{L})^{-1}$, and $\tilde{f}(\xi) = f(\xi) + \xi$ for $\xi \in \mathbf{R}$. For simplicity, write φ_u by φ . Then μ_u is the principal eigenvalue of $\tilde{L} - \tilde{f}'(u)$ and φ is an eigenfunction of $\tilde{L} - \tilde{f}'(u)$ corresponding to μ_u . We set $E_1 = \{s\varphi: s \in \mathbf{R}\}$. Then there is a closed subspace E_2 of $L^\infty(Q_T)$ which is invariant under $\tilde{K} \circ \tilde{f}'(u)$ such that $L^\infty(Q_T) = E_1 \oplus E_2$. Denoting by P_i the projection from $L^\infty(Q_T)$ onto E_i for $i = 1, 2$, P_i is a bounded operator, that is, there exists $M > 0$ satisfying

$$\|P_i v\|_\infty \leq M \|v\|_\infty, \quad i = 1, 2,$$

for all $v \in L^\infty(Q_T)$. Since \tilde{f} is a mapping of class C^2 from $L^\infty(Q_T)$ into itself, the following Taylor expansions hold in $B_\delta(0)$ for some $\delta > 0$:

$$(1) \quad \tilde{f}(u + v) = \tilde{f}(u) + \tilde{f}'(u)v + \frac{f''(u)}{2}v^2 + h_1(v),$$

$$(2) \quad \tilde{f}'(u + v) = \tilde{f}'(u) + f''(u)v + h_2(v), \quad f''(u + v) = f''(u) + h_2(v)$$

for $v \in B_\delta(0)$, where $h_1(v) \in o(\|v\|_\infty^2)$, $h_2(v) \in o(\|v\|_\infty)$, and $h_3(v) \in O(\|v\|_\infty^{-2})$. Here we choose $\varepsilon > 0$ satisfying

$$\{s\varphi: |s| < \varepsilon\} \times \{v \in E_2: \|v\| < \varepsilon\} \subset B_\delta(0)$$

and put $U_\varepsilon(u) = u + \{s\varphi: |s| < \varepsilon\} \times \{v \in E_2: \|v\| < \varepsilon\}$.

Now we show that $(f''(u)/\tilde{f}'(u))\varphi^2 \notin E_2$. From the definition of f and \tilde{f} , we have that $f''(u)/\tilde{f}'(u) \geq 0$ and $(f''(u)/\tilde{f}'(u))\varphi^2 \in E$. Assume that $(f''(u)/\tilde{f}'(u))\varphi^2 \in E_2$. Then we have that there is $w \in L^\infty(Q_T)$ such that

$$(3) \quad \tilde{L}w - f'(u)w = \tilde{f}''(u)/\tilde{f}'(u)\varphi^2.$$

Since 0 is the principal eigenvalue of $\tilde{L} - \tilde{f}'(u)$, we obtain that there exists a positive function $\tilde{\varphi} \in E$ satisfying

$$(4) \quad -\tilde{\varphi}_t - \Delta\tilde{\varphi} + \tilde{\varphi} - \tilde{f}'(u)\tilde{\varphi} = 0$$

by [4]. From (3) and (4), it follows that

$$\begin{aligned} 0 &< \int_0^T \int_\Omega \frac{f''(u)}{\tilde{f}'(u)} \varphi^2 \tilde{\varphi} dx dt \\ &= \int_0^T \int_\Omega (w_t - \Delta w + w - \tilde{f}'(u)w) \tilde{\varphi} dx dt \\ &= \int_0^T \int_\Omega (-\tilde{\varphi}_t - \Delta\tilde{\varphi} + \tilde{\varphi} - \tilde{f}'(u)\tilde{\varphi}) w dx dt = 0. \end{aligned}$$

This contradiction implies that $(f''(u)/\tilde{f}'(u)) \notin E_2$. Therefore, there exists $r \neq 0$ satisfying

$$\frac{f''(u)}{\tilde{f}'(u)} \varphi^2 - P_2 \left(\frac{f''(u)}{\tilde{f}'(u)} \varphi^2 \right) = r\varphi.$$

Fix $v \in E_2$ with $\|v\| < \varepsilon$, and define

$$\psi_v(s) = \langle \tilde{L}(u + v + s\varphi) - \tilde{f}(u + v + s\varphi) - h, r\varphi \rangle$$

for $s \in \mathbf{R}$ with $|s| < \varepsilon$. Then we have from (1) and (2) that

$$\begin{aligned} \psi_v(0) &= \left\langle \tilde{L}v - \tilde{f}'(u)v - \frac{f''(u)}{2}v^2 - h_1(v), r\varphi \right\rangle, \\ \psi'_v(0) &= -\langle f''(u)v\varphi + h_2(v)\varphi, r\varphi \rangle, \end{aligned}$$

and

$$\psi''_v(0) = -\langle f''(u)\varphi^2 + h_3(v)\varphi^2, r\varphi \rangle.$$

Consequently, we find

$$\begin{aligned} (5) \quad \psi_v(s) &= \left\langle \tilde{L}v - \tilde{f}'(u)v - \frac{f''(v)}{2}v^2 - h_1(v), r\varphi \right\rangle \\ &\quad - \langle f''(u)v\varphi + h_2(v)\varphi, r\varphi \rangle s \\ &\quad - \frac{1}{2} \langle f''(u)\varphi^2 + h_3(v)\varphi^2, r\varphi \rangle s^2 + h_4(s), \end{aligned}$$

where $h_4(s) \in o(s^2)$. For $\rho \in [0, 1]$, set

$$g_\rho(t, x, \xi) = g(t, x, \xi) + \rho r \tilde{f}'(u)\varphi.$$

We claim that for each $0 < \rho \leq 1$, the compact mapping $K \circ g_\rho$ has no fixed point in $U_\varepsilon(u)$ and that u is the unique fixed point of $K \circ g$ in $U_\varepsilon(u)$. Suppose that

$$(6) \quad L(u + w + \tau\varphi) = g_\rho(t, x, u + w + \tau\varphi)$$

for some $\rho \in [0, 1]$, $w \in E_2$, and $\tau \in \mathbb{R}$ with $\|w\|, |\tau| < \varepsilon$. Then we get by (1) that

$$\tilde{L}w - \tilde{f}'(u)w - \frac{f''(u)}{2}(w + \tau\varphi)^2 - h_1(w + \tau\varphi) = \rho r \tilde{f}'(u)\varphi.$$

Multiplying this equality by $1/\tilde{f}'(u)$ and projecting both sides of the equality onto E_2 , it follows that

$$(7) \quad \tilde{L}w - \tilde{f}'(u)w = \frac{\tilde{f}'(u)}{2}P_2\left(\frac{f''(u)}{\tilde{f}'(u)}(w + \tau\varphi)^2\right) + \tilde{f}'(u)P_2\left(\frac{h_1(w + \tau\varphi)}{\tilde{f}'(u)}\right)$$

and hence

$$w = (\tilde{L} - \tilde{f}'(u))^{-1} \left(\frac{\tilde{f}'(u)}{2}P_2\left(\frac{f''(u)}{\tilde{f}'(u)}(w + \tau\varphi)^2\right) + \tilde{f}'(u)P_2\left(\frac{h_1(w + \tau\varphi)}{\tilde{f}'(u)}\right) \right).$$

From the inequality above, we have that there exists $C_1 > 0$ satisfying

$$\|w\|_\infty \leq C_1(\|w\|_\infty^2 + \tau^2).$$

Since we may choose ε so small that $C_1\varepsilon < 1/2$, we obtain from the inequality above that

$$(8) \quad \|w\|_\infty \leq 2C_1\tau^2$$

for each $w \in E$ satisfying (6). Then we have from (5), (6), (7), and the equality above that

$$\begin{aligned} 0 &\leq \rho \langle \tilde{f}'(u)r\varphi, r\varphi \rangle = \psi_w(\tau) \\ &= -\frac{1}{2} \left\langle \left\langle f''(u)\varphi^2 - \tilde{f}'(u)P_2\left(\frac{f''(u)}{\tilde{f}'(u)}\varphi^2\right), r\varphi \right\rangle \right\rangle \tau^2 + h_5(\tau) \\ &= -\frac{1}{2} \langle \tilde{f}'(u)r\varphi, r\varphi \rangle \tau^2 + h_5(\tau), \end{aligned}$$

where $h_5 \in o(\tau^2)$. Then we have from the inequality above that $w \in E_2$ with $\|w\| \leq \varepsilon$ satisfies (6) only when $\rho = 0$ and $w = 0$. Thus we have shown that $K \circ g_\rho$ has no fixed point in $U_\varepsilon(u)$ for $1 < \rho \leq 1$ and u is the unique fixed point of $K \circ g$ in $U_\varepsilon(u)$. Then according to the homotopy invariance of the Leray-Schauder degree, we find that

$$\deg(I - K \circ g, U_\varepsilon(u), 0) = \deg(I - K \circ g_1, U_\varepsilon(u), 0) = 0.$$

This completes the proof.

Proof of Theorem. By Lemma 1, there is $r > 0$ sufficiently small such that the problem (P) possesses a solution u_1 in $B_r(0)$. We may assume that $0 \leq f'(v) < \lambda_1$ on Ω for each $v \in B_r(0)$. Letting μ_{u1} and φ_{u1} as before, it holds that

$$(\varphi_{u1})_t = \Delta\varphi_{u1} - f'(u)\varphi_{u1} = \mu_{u1}\varphi_{u1}.$$

Assume that $\mu_{u_1} \leq 0$. From this equality and the definition of λ_1 , it follows that

$$\int_0^T \int_{\Omega} \lambda_1 \varphi_1 \varphi_{u_1} dx dt = \int_0^T \int_{\Omega} \{f'(u) + \mu_{u_1}\} \varphi_{u_1} \varphi_1 dx dt < \int_0^T \int_{\Omega} \lambda_1 \varphi_{u_1} \varphi_1 dx dt.$$

This contradiction implies that $\mu_{u_1} > 0$, and then u_1 is stable. On the other hand, taking R sufficiently large, we have by Lemma 2 that

$$\deg(I - K \circ g, B_R(0) \setminus B_r(0), 0) = -1.$$

We can see from the proof of Lemma 3 that if u is a solution of (P) in $B_R(0)$ satisfying $\mu_u \geq 0$, then u is isolated. Then by Lemma 3, we conclude that there exists $u_2 \in B_R(0) \setminus B_r(0)$ with $\mu_{u_2} < 0$. Therefore, u_2 is unstable. This completes the proof.

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