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ON THE COVERING AND THE ADDITIVITY NUMBER OF THE REAL LINE

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ABSTRACT. We show that the real line R cannot be covered by k many nowhere dense sets iff *whenever* $D = \{D_i : i \in k\}$ *is a family of dense open sets of* R *there exists a countable dense set* G *of* R *such that* $|G \backslash D_i| < \omega$ *for all* $i \in k$. We also show that the union of k meagre sets of the real line is a meagre set iff *for every family* $D = \{D_i : i \in k\}$ *of dense open sets of* R *and for every countable dense set* G *of* R *there exists a dense set* $Q \subseteq G$ *such that* $|Q \backslash D_i| < \omega$ *for all* $i \in k$.

1. NOTATION AND TERMINOLOGY

The notation and terminology which we will use is standard and can be found in [8] or [9]. In particular, if A, B are sets and λ any cardinal finite or infinite, then $[A]^{\lambda}$, $[A]^{<\lambda}$, and $[A]^{\leq\lambda}$ denote the sets of all subsets of A of size λ, $<\lambda$, and $\leq \lambda$ respectively. ω^{ω} denotes the Baire space, i.e., the set of all functions ${}^{\omega}\omega$ together with the topology, having as a base the collection of all (clopen) sets of the form

$$[p] = \{f \in {}^{\omega}\omega : p \subseteq f\}, \qquad p \in \omega^{<\omega} = \bigcup\{{}^{n}\omega : n \in \omega\}.$$

We say that the family $A \subseteq [\omega]^{\omega}$ has the *countable π-base property* (cπbp), iff there exists a $B \in [[\omega]^{\omega}]^{\omega}$, call it a *$\pi$-base*, such that

$$(\forall a \in A)(\forall b \in B)(\exists d \in B)(d \subseteq a \cap b).$$

In order to avoid confusion, let us remark that the notions of cπbp and a *filter* A of $[\omega]^{\omega}$ having a *countable base* are not the same. B need not be included in A. In fact, B may contain disjoint sets.

$A \subseteq [\omega]^{\omega}$ has the *strong finite intersection property* (sfip), iff $|\bigcap Q| = \omega$ for every $Q \in [A]^{<\omega}$.

We say that a set $S \in [\omega]^{\omega}$ is an *infinite pseudointersection* of the family $A \subseteq [\omega]^{\omega}$ iff $(\forall a \in A)(|S \backslash a| < \omega)$.

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Let k, m be any infinite cardinal numbers less than the power of the continuum $\mathfrak{c}$. We define the following combinatorial statements:

$\mathrm{PS}(k) \equiv$ Every $A \in [[\omega]^\omega]^{\leq k}$ with the $c\pi$bp has an infinite pseudointersection.

We remark here that it is consistent with ZFC that there exist sets $A \subseteq [\omega]^\omega$ having the sfip but no countable π-base (there are filters having no countable base). Indeed, let $(M, \in)$ be a model of ZFC such that $M \vDash \mathfrak{p} = \omega_1 + \mathrm{cov}(R) = \omega_2$ (see [9, Theorem 3]), where $\mathfrak{p}$ is the *pseudointersection number* and $\mathrm{cov}(R)$ the *covering number* of the real line. Then there exists a set $A \subseteq [\omega]^\omega$, $|A| = \omega_1$, having the sfip but no infinite pseudointersection. If $B = \{b_n : n \in \omega\} \subseteq [\omega]^\omega$ were a π-base for A, then, using $\mathrm{cov}(R) > \omega_1$, one can easily construct a pseudointersection S for A which is a contradiction.

$\mathrm{SPS}(k) \equiv$ For every $A \in [[\omega]^\omega]^{\leq k}$ and for every π-base $B \in [[\omega]^\omega]^\omega$ for A there exists an infinite pseudointersection S of A meeting each member of B is an infinite set.

$\mathrm{Equal}(k) \equiv (\forall F \in [{}^\omega\omega]^{\leq k})(\exists h \in^\omega \omega)(\forall f \in F)(\exists^\infty n)(f(n) = h(n))$, where $\exists^\infty n$ abbreviates the statement "there are infinitely many".

$\mathrm{Equal}^*(k) \equiv (\forall F \in [{}^\omega\omega]^{\leq k})(\exists h_n \in^\omega \omega)(\forall f \in F)(\forall^\infty n)(\exists m \in \omega)(f(n) = h_n(n))$, where $\forall^\infty n$ abbreviates the statement "for all but finitely many".

$\mathrm{Bounded}(k) \equiv (\forall F \in [{}^\omega\omega]^{\leq k})(\exists h \in^\omega \omega)(\forall f \in F)(\forall^\infty n)(f(n) \leq h(n))$.

$\mathrm{Weak\ Bounded}(k) \equiv (\forall F \in [{}^\omega\omega]^{\leq k})(\exists h \in^\omega \omega)(\forall f \in F)(\exists^\infty n)(f(n) < h(n))$.

The *bounding number* $\mathfrak{b}$ and the *dominating number* $\mathfrak{d}$ are the least cardinal numbers k for which the statements $\mathrm{Bounded}(k)$ and $\mathrm{Weak\ Bounded}(k)$ fail respectively.

Let (X, T) be a topological space. A dense subset Q of the poset $(T \backslash (\varnothing), \subseteq)$ is called a *π-base for* X.

A set $N \subseteq X$ is called *nowhere dense* iff $\mathrm{int}(\overline{N}) = \varnothing$, and a set $A \subseteq X$ is called *meagre* iff A is the union of countable many nowhere dense sets. The *covering number*, $\mathrm{cov}(X)$, and the *additivity number*, $\mathrm{add}(X)$, of the space X are given by:

$\mathrm{cov}(X) \equiv \min\{|D| : D$ is a family of dense open sets in X with $\bigcap D = \varnothing\}$,

$\mathrm{add}(X) \equiv \min\{|D| : D$ is a family of meagre sets of X but $\bigcup D$ is not a meagre set of $X\}$ respectively.

All undefined terms are used as in [5, 8, 10].

2. Introduction and preliminary results

The covering number, $\mathrm{cov}(R)$, of the real line may be a singular cardinal number [11]. In this case, it has been shown by A. W. Miller in [13] that $\mathrm{cov}(R)$ has uncountable cofinality, and T. Bartoszynski and J. I. Ihoda have shown in [3] that its cofinality is greater than or equal to $\mathrm{add}(\mathbf{L})$, where $\mathrm{add}(\mathbf{L})$ is the additivity number of the ideal $\mathbf{L}$ of all Lebesque null sets of the real line. It is an open question (see [3, 4, 15]) whether $\mathrm{cf}(\mathrm{cov}(R))$ *can be less than* $\mathrm{add}(R)$. Note that if $\mathrm{cov}(R) \leq \mathfrak{b}$, then by Lemma 3 we have $\mathrm{cov}(R) = \mathrm{add}(R)$ and that if $\mathrm{cov}(R) = \mathfrak{d}$, then $\mathrm{cf}(\mathfrak{d}) \geq \mathfrak{b}$ (see [5, Theorem 3.1(d), p. 116]). Thus the above-mentioned question is nontrivial only in case $\mathfrak{b} < \mathrm{cov}(R) < \mathfrak{d}$ (it is known (see [7]) that $\mathrm{cov}(R) \leq \mathfrak{d}$).

The technology that exists in this area seems to be inadequate in answering the above-mentioned question. The aim of this paper is mainly to find new characterisations of the cardinals $\mathrm{cov}(R)$ and $\mathrm{add}(R)$ and then reformulate the problem in terms of these new characterisations. Such characterisations may be raised from simple combinatorial statements holding true in the presence of the continuum hypothesis **CH**; e.g.,

For every family $\mathbf{D}$, $|\mathbf{D}| < \mathfrak{c}$, *of dense open sets of the real line* R *there exists a countable dense set* $G \subseteq R$ *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.

For every family $\mathbf{D}$, $|\mathbf{D}| < \mathfrak{c}$, *of dense open sets of the real line* R *and for every countable dense set* $Q \subseteq R$ *there exists a countable infinite set* $G \subseteq Q$ *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.

For every family $\mathbf{D}$, $|\mathbf{D}| < \mathfrak{c}$, *of dense open sets of the real line* R *and every countable dense set* $Q \subseteq R$ *there exists a countable dense set* $G \subseteq Q$ *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.

Let us recall some characterisations of $\operatorname{cov}(R)$ and $\operatorname{add}(R)$ which we will use in the sequel.

Lemma 1 ([1], Miller-Bartoszynski). $\operatorname{cov}(R) > k$ *iff* $\mathrm{Equal}(k)$.

Lemma 2 ([6]). $\operatorname{add}(R) > k$ *iff for every family* $D = \{D_i : i \in k\}$ *of dense open sets of* R *there exists a family* $Q = \{Q_n : n \in \omega\}$ *of dense open sets of* R *such that for every* $i \in k$ *there exists* $n \in \omega$ *with* $Q_n \subseteq D_i$.

Lemma 3 ([14], Miller-Truss). $\operatorname{add}(R) > k$ *iff* $\mathrm{Bounded}(k)$ *and* $\operatorname{cov}(R) > k$.

As a corollary to Exercise C4 from [10, p. 242], one can easily establish the following lemma.

Lemma 4. *Let* $(P, \leq)$ *and* $(Q, \leq)$ *be any two countable nonatomic posets. Then there exist dense sets* $P' \subseteq P$ *and* $Q' \subseteq Q$ *such that* $(P', \leq)$ *and* $(Q', \leq)$ *are isomorphic as posets. (There exists a bi-injective function* $H: P' \to Q'$ *such that* $H(p) \leq H(q)$ *iff* $p \leq q$ *for all* $p, q \in P'$.)

3. Characterisations of $\operatorname{cov}(R)$

Theorem 1. *The following are equivalent for every cardinal* k :

(1) $\operatorname{cov}(R) > k$.
(2) $\mathrm{Equal}(k)$.
(3) *For every family* $\mathbf{D}$, $|\mathbf{D}| < k$, *of dense open sets of the real line* R *there exists a countable dense set* $G \subseteq R$ *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.
(4) *For every family* $\mathbf{D}$, $|\mathbf{D}| < k$, *of dense open sets of the real line* R *there exists a countable infinite set* G *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.
(5) $\mathrm{Equal}^*(k)$.
(6) $\mathrm{PS}(k)$.
(7) MA_k *(countable) (Martin's Axiom restricted to countable posets).*
(8) *For every family* $\mathbf{D}$, $|\mathbf{D}| < k$, *of dense open sets of the real line* R *and for every countable dense set* $Q \subseteq R$ *there exists a countable infinite set* $G \subseteq Q$ *such that* $|G \backslash D| < \omega$ *for all* $D \in \mathbf{D}$.
(9) *For every* T_1 *space* X *of countable* π*-weight and every family* $D = \{D_i : i \in k\}$ *of dense open sets in* X *there exists a countable infinite set* $G \subseteq X$ *such that* $|G \backslash D_i| < \omega$ *for all* $i \in k$.

Proof. $(1) \leftrightarrow (2)$ and $(1) \leftrightarrow (7)$ are well known. $(1) \leftrightarrow (2)$ is Lemma 1, and $(1) \leftrightarrow (7)$ is a consequence of Lemma 4.

$(1) \to (3)$ and $(3) \to (4)$ are straightforward.

(4) → (5). Fix $F = \{f_i : i \in k\} \subseteq^{\omega} \omega$. For every $i \in k$, $D_i = \{f \in^{\omega} \omega : f_i(n) = f(n) \text{ for some } n \in \omega\}$ is clearly a dense open subset of ${}^{\omega}\omega$. Using (4), Lemma 4, and standard density arguments one can easily verify that there exists a countable infinite set $G \subseteq {}^{\omega}\omega$ such that $|G \backslash D| < \omega$ for all $i \in k$. It is not hard to see that G satisfies Equal*(k) for the collection F as required.

(5) → (2). First we need to show

Claim. Equal*(k) → Weak Bounded(k).

Proof. Let $F \subseteq \omega^{\omega}$ be a family of size k. Using the assumption choose a family $\{g_n : n \in \omega\}$ such that

$$(\forall f \in F)(\forall^{\infty} n)(\exists \kappa)(g_n(\kappa) = f(\kappa)).$$

By passing to a subsequence we can assume that, for every n, $g_n(n)$, $g_{n+1}(n)$, $g_{n+2}(n)$, ... are all equal or pairwise different.

Subclaim. For every $f \in F$ there exists n and $\kappa \leq 2n$ such that $g_\kappa(n) = f(n)$.

Proof. Suppose not, and let $f \in F$ be such that $g_\kappa(n) \neq f(n)$ for $\kappa \leq 2n$. Find κ_0 and a sequence $\{n_\kappa : \kappa \geq \kappa_0\}$ such that $g_\kappa(n_\kappa) = f(n_\kappa)$ for $\kappa \geq \kappa_0$. By the assumption, $\kappa > 2n_\kappa$. Consider terms n_{κ_0}, n_{κ_0+1}, ..., $n_{2\kappa_0}$. Note that all these terms are smaller than κ_0. Thus, for some $i < j$, $n_{\kappa_0+i} = n_{\kappa_0+j} = n^*$. It follows that

$$g_{\kappa_0+i}(n^*) = g_{\kappa_0+j}(n^*) = f(n^*).$$

In particular, $g_{n^*}(n^*) = f(n^*)$, a contradiction finishing the proof of the subclaim.

To finish the proof of the claim define

$$g(n) = \max\{g_\kappa(n) : \kappa \leq 2n\} + 1 \quad \text{for } n \in \omega.$$

Clearly

$$(\forall f \in F)(\exists n)(f(n) < g(n)).$$

This shows that F cannot be dominating as required.

To complete the proof of (5) → (2), fix $F = \{f_i : i \in k\} \subseteq {}^{\omega}\omega$ and let $A = \{A_j : j \in \omega\} \subseteq [\omega]^{\omega}$ be a partition of ω. For every $j \in \omega$ let $G_j = \{g(j, n) : n \in \omega\}$ satisfy Equal*(k) for the collection $F_j = \{f_i | A_j : i \in k\}$. Define a function $h_j : \omega \to \omega$ by requiring

$$h_i(j) = \min(\{m : (\forall l \geq m)(\exists u \in A_j)(g(j, l)(u) = f_i(u))\}).$$

Let, by the claim, $h: \omega \to \omega$ satisfy Weak Bounded(k) for the collection $H = \{h_i : i \in k\}$, and define a function $g: \omega \to \omega$ by letting

$$g/A_j = g(j, h(j)).$$

Clearly g satisfies Equal(k) for the collection F as required.

(7) → (6). Fix a set $A = \{A_i : i \in k\} \subseteq [\omega]^{\omega}$, and let $B = \{b_n : n \in \omega\} \subseteq [\omega]^{\omega}$ be a π-base for A. Clearly $D_i = \{b \in B : b \subseteq A_i\}$ is dense in $(B, \subseteq)$ for all $i \in k$. Using (7), one can easily verify that there exists a filter $G = \{b_n : n \in \omega\}$ of $(B, \subseteq)$ meeting each D_i nontrivially. Via an easy induction pick for every $n \in \omega$,

$$s_n \in (b_n \backslash \{s_m : m \in n\}).$$

Clearly, $S = \{s_n : n \in \omega\}$ is an infinite pseudointersection for A as required.

(6) → (8). Let $D = \{D_i : i \in k\}$, Q, and B be a family of dense open sets, a countable dense set, and a countable base for R, respectively.

Put

$$A_i = D_i \cap Q \text{ for all } i \in k$$

and

$$B_b = b \cap Q \text{ for all } b \in B.$$

Clearly $B^* = \{B_b : b \in B\}$ is a π-base for $A = \{A_i : i \in k\}$. Thus, by (6), there exists an infinite pseudointersection $G \subseteq Q$ for A. This certainly implies that $|G \backslash D_i| < \omega$ for all $i \in k$ as required.

(8) → (4) and (6) → (9) are straightforward.

To finish the proof of the theorem it suffices to show

(9) → (6). Fix $A = \{A_i : i \in k\} \subseteq [\omega]^\omega$, and let $B = \{B_n : n \in \omega\} \subseteq [\omega]^\omega$ be a π-base for A. Without loss of generality we may assume that $\bigcup A = \bigcup B = \omega$. For every $x, n \in \omega$, we let $B_n(x) = B_n \backslash \{x\}$ and put

$$F = \{B_n(x) : n, x \in \omega\}.$$

Clearly the topology $\mathbf{T}_F$ which is produced from the subbase F is T_1 and second countable. Furthermore, A is a family of dense open sets in the topological space $(\omega, \mathbf{T}_F)$. Thus by (9) there exists a set $G \subseteq X$, $|G| = \omega$, such that $|G \backslash A_i| < \omega$ for all $i \in k$ as required. □

As an easy corollary of Theorem 1(4) we have $\mathrm{cf}(\mathrm{cov}(R)) > \omega$. In fact, more than that is true. Namely, let $\mathbf{s}_c$ denote the least k for which the following statement fails.

For every family $G \subseteq {}^\omega\omega$, $|G| < \mathfrak{d}$, and every family $F = \{f_i : i \in k\} \subseteq {}^\omega\omega$ there exists a family $H = \{h_n : n \in \omega\} \subseteq {}^\omega\omega$ such that for every $g \in G$ if g meets all but less than k many members of F, then g meets all but finitely many members of H

Then, we have:

Corollary 1. *If* $\mathrm{add}(R) \leq \mathbf{s}_c$, *then* $\mathrm{cf}(\mathrm{cov}(R)) \geq \mathrm{add}(R)$.

Proof. Assume on the contrary, and let

$$\mathrm{cf}(\mathrm{cov}(R)) = \lambda < \mathrm{add}(R) \leq \mathbf{s}_c.$$

In view of [5, Theorem 3.1(d), p. 116], we may assume that $k = \mathrm{cov}(R) < \mathfrak{d}$. Fix, by Theorem 1, a family

$$G = \{g_i : i \in k\} \subseteq {}^\omega\omega$$

such that

$$(\forall H \in [{}^\omega\omega]^\omega)(\exists i \in k)(\exists^\infty h \in H)(h \cap g_i = \varnothing).$$

Fix $\{\lambda_j : j \in \lambda\}$ a cofinal set in k. For every $j \in \lambda$ let $f_j : \omega \to \omega$ be such that

$$(\forall i \in \lambda_j)(\exists^\infty n)(g_i(n) = f_j(n)).$$

Put $F = \{f_j : j \in \lambda\}$, and note that

$$(\forall i \in k)((\forall^\infty j \in \lambda)(f_j \cap g_i \neq \varnothing)).$$

Fix, by $\mathbf{s}_c > \lambda$, a set $H \in [{}^\omega\omega]^\omega$ satisfying

$$(\forall i \in k)(\forall^\infty h \in H)(h \cap g_i \neq \varnothing).$$

This contradicts the choice of G and finishes the proof. □

4. Characterisations of $\mathrm{add}(R)$

Theorem 2. *The following are equivalent for every cardinal* k :

(1) $\mathrm{add}(R) > k$.
(2) $\mathrm{SPS}(k)$.
(3) *For every family* $\mathbf{D}$, $|\mathbf{D}| < k$, *of dense open sets of the real line* R *and every countable dense set* $Q \subseteq R$ *there exists a countable dense set* $G \subseteq Q$ *such that* $|G\backslash D| < \omega$ *for all* $D \in \mathbf{D}$.
(4) $\mathrm{Equal}^*(k)$ *and* $\mathrm{Bounded}(k)$.
(5) $\mathrm{Equal}(k)$ *and* $\mathrm{Bounded}(k)$.
(6) MA_k *(countable) and* $\mathrm{Bounded}(k)$.
(7) *For every* T_1 *space* X *of countable* π*-weight, every family* $D = \{D_i : i \in k\}$ *of dense open sets in* X, *and every countable dense set* $Q \subseteq X$ *there exists a dense set* $G \subseteq Q$ *such that* $|G\backslash D_i| < \omega$ *for all* $i \in k$.

Proof. $(1) \to (3)$ follows immediately from Lemma 2.

$(3) \to (4)$. In view of Theorem 1 and Lemma 3, we only have to show that (3) implies $\mathrm{Bounded}(k)$. Fix $F = \{f_i : i \in k\} \subseteq {}^{\omega}\omega$. Without loss of generality we may assume that each f_i is a strictly increasing function. It is easy to see that

$$\mathbf{D}_i = \{f \in {}^{\omega}\omega : f(n) > f_i(n) \text{ for some } n \in \omega\}$$

is a dense open set of the Baire space ω^{ω}. Let

$$G = \{g_n : n \in \omega\} \subseteq^{\omega} \omega,$$

where g_n is eventually equal to zero. Clearly G is dense in ω^{ω}. By (3), Lemma 4, and standard density arguments, there exists a dense set $Q = \{q_n : n \in \omega\} \subseteq G$ such that

$$|Q\backslash D_i| < \omega \quad \text{for all } i \in k.$$

Choose a subsequence $Q' = \{q_{n_v} : v \in \omega\}$ of Q such that

$$(\forall u \geq n_v)(q_{n_v}(u) = 0) \wedge (\forall v \in \omega)((q_{n_{v+1}} \,|\, n_v) = 0).$$

On the basis of Q' we define a function $f\colon \omega \to \omega$ by requiring

$$f(u) = \max\{q_{n_v}(t) : v \leq u+1,\ t \in \mathrm{Dom}(q_{n_v})\}.$$

f dominates F. Indeed, fix $i \in k$, and let $v' \in \omega$ satisfy

$$(\forall v \geq v')(q_{n_v} \in D_i).$$

This means that

$$(\forall v \geq v')(\exists n \in [n_{v-1}, n_v))(q_{n_v}(n) > f_i(n)).$$

Thus, if $v > v'$, then we have

$$f(u) \geq \max\{q_{n_{u+1}}(n) : n \in [n_u, n_{u+1})\} > f_i(n_u) > f_i(u),$$

and the desired result follows.

Implications $(4) \to (5) \to (6) \to (1)$ are clear (Theorem 1 and Lemma 3).

$(6) \to (2)$. Fix $A = \{A_i : i \in k\} \subseteq [\omega]^{\omega}$, and let $B = \{b_n : n \in \omega\} \subseteq [\omega]^{\omega}$ be a base for A. For every $n \in \omega$, by (6), fix $S_n = \{s(n, m) : n \in \omega\} \subseteq b_n$ an infinite pseudointersection for A. For every $i \in k$ define a function $f_i\colon \omega \to \omega$ by requiring

$$f_i(n) = \min\{v : s(n, m) \in D_i \text{ for all } m \geq v\}.$$

Put $F = \{f_i : i \in k\}$. By Lemma 3 $(k < \mathrm{add}(R) \leq \mathfrak{b})$, there exists a function $f: \omega \to \omega$ such that

$$(\forall i \in k)(\forall^\infty n)(f_i(n) \leq f(n)).$$

On the basis of f we define a set S by letting

$$S = \{s(n, m) : n \in \omega, \ m \geq f(n)\}.$$

It can be readily verified that S is a pseudointersection of A meeting every member of B in an infinite set as required.

(2) $\to$ (3). Fix $D = \{D_i : i \in k\}$ and G as in (3). By Lemma 2, there is a family $Q = \{Q_n : n \in \omega\}$ of dense open sets of R such that

$$(\forall i \in k)(\exists n \in \omega)(Q_n \subseteq D_i).$$

Let $B = \{B_n : n \in \omega\}$ be a base for the topology of R. By induction, choose a set

$$T = \{t_n : n \in \omega\} \subseteq G$$

such that

$$t_n \in \left(\left(B_n \cap \left(\bigcap\{Q_m : m \leq n\}\right) \cap G\right) \setminus \{t_m : m < n\}\right).$$

Clearly T is dense in R, and $|T \setminus Q_n| < \omega$ for all $n \in \omega$. This, when combined with the above, finishes the proof of (2) $\to$ (3).

(7) $\to$ (3) is straightforward.

(6) $\to$ (7). It suffices in view of Theorem 1 to show that $\mathrm{PS}(k)$ and Bounded(k) together imply (2). This can be established as in (6) $\to$ (2) of the present theorem. □

Let $\mathrm{PS}(\omega_1, k)$ be the generalisation of $\mathrm{PS}(k)$ to the next higher cardinal with the additional requirement that B be a countable closed base ($\bigcap Q \in B$ for every $Q \in [B]^\omega$) of size ω_1. Of course the continuum hypothesis must be assumed here. Also, let $\omega^* = \beta\omega \mid \omega$ denote the remainder of the Stone-Čech compactification of ω with the discrete topology.

Question 1. Assume **CH**. Does $\mathrm{PS}(\omega_1, k)$ imply $\mathrm{cov}(\omega^*) > k$?

Question 2. Can $\mathfrak{s}_c$ be strictly less than $\mathrm{add}(R)$?

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References

1. T. Bartoszynski, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **81** (1984), 209–213.
2. ——, *Combinatorial aspects of measure and category*, Fund. Math. **127** (1987), 225–239.
3. T. Bartoszynski and J. I. Ihoda, *On the cofinality of the smallest covering of the real line by meagre sets*, J. Symbolic Logic **54** (1989), 828–832.
4. T. Bartoszynski, J. I. Ihoda, and S. Shelah, *The cofinality of cardinal invariants related to measure and category*, J. Symbolic Logic **54** (1989), 719–726.

5. E. K. van Douwen, *The integers and topology*, Handbook of Set Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 113–167.
6. D. H. Fremlin, *The partially ordered sets of measure and category and Tukey's ordering*, preprint.
7. ——, *Cichon's diagram*, Seminaire d'Initiation a l'Analyse (G. Choque, M. Rogalski, and J. Saint Raymond, eds.), 23eme anne: 1983–1984, Université Pierre et Marie Curie, (Paris-VI), Paris, expose 5.
8. T. Jech, *Set theory*, Academic Press, New York, 1978.
9. K. Keremedis, *Partition reals and the consistency of* $t < \operatorname{add}(R)$, Math. Log. Quart. **39** (1993).
10. K. Kunen, *Set theory*, North-Holland, Amsterdam, 1983.
11. ——, *Random and Cohen reals*, Handbook of Set Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 887–911.
12. A. W. Miller, *A characterisation of the least cardinal number for which the Baire category theorem fails*, Proc. Amer. Math. Soc. **86** (1982), 498–502.
13. ——, *The Baire category theorem and cardinals of countable cofinality*, J. Symbolic Logic **47** (1982), 275–288.
14. ——, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), 93–114.
15. J. E.Vaughan, *Small uncountable cardinals and topology*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990.

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