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OSCILLATION AND NONOSCILLATION CRITERIA FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation and nonoscillation criteria for the first-order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \ \tau(t) < t,$$

are established in the case where

$$\int_{\tau(t)}^{t} p(s)\,ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = \frac{1}{e}.$$

1. INTRODUCTION

The qualitative properties of the solutions of the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $\tau(t) < t$, have been the subject of many investigations. The first systematic study was made by Myshkis [6]. Among others he has shown [5] that all solutions of (1) oscillate if

$$p(t) \geq 0, \qquad \limsup_{t\to\infty}[t - \tau(t)] < \infty, \qquad \liminf_{t\to\infty}[t - \tau(t)] \cdot \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

Later these conditions were improved, by Ladas [4] and Koplatadze and Chanturija [3], to

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}. \tag{2}$$

Concerning the constant $\frac{1}{e}$ in (2) we mention that if the inequality

$$\int_{\tau(t)}^{t} p(s)\,ds \leq \frac{1}{e}$$

holds, then, according to a result in [3], (1) has a nonoscillatory solution. To the best of our knowledge there is no result in the case when we have

$$\int_{\tau(t)}^{t} p(s)\,ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = \frac{1}{e}. \tag{3}$$

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In connection with the delay function $\tau(t)$ in (1) we suppose that $\tau(t)$ is strictly increasing on $[t_0, \infty)$, $\lim_{t\to\infty} \tau(t) = \infty$, and its inverse is $\tau_{-1}(t)$ $(\tau_{-1}(t) > t)$. Let $\tau_{-k}(t)$ be defined on $[t_0, \infty)$ by

$$\tau_{-k-1}(t) = \tau_{-1}(\tau_{-k}(t)) \quad \text{for } k = 1, 2, \ldots,$$

and let

$$t_k = \tau_{-k}(t_0), \qquad k = 1, 2, \ldots. \tag{4}$$

Clearly $t_k \to \infty$ as $k \to \infty$.

The coefficient $p(t)$ is assumed to be a piecewise continuous function and satisfies the relation

$$\int_{\tau(t)}^{t} p(s)\, ds \geq \frac{1}{e}.$$

Let $\varphi(t)$ be a continuous function on $[\tau(t_0), t_0]$. A function $x(t)$ is a solution of (1), associated with the initial function $\varphi(t)$, if $x(t) = \varphi(t)$ on $[\tau(t_0), t_0]$, $x(t)$ is continuous on $[\tau(t_0), \infty)$, is differentiable almost everywhere on (t_0, ∞), and satisfies (1).

As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory*.

Among the functions $p(t)$ we define a set $\mathscr{A}_\lambda$ for $0 < \lambda \leq 1$ as follows.

Definition. The piece-wise continuous function $p(t) : [t_0, \infty] \to [0, \infty]$ belongs to $\mathscr{A}_\lambda$ if

$$\int_{\tau(t)}^{t} p(s)\, ds \geq \frac{1}{e}, \qquad t \geq t_1,$$

(5)

$$\int_{\tau(t)}^{t} p(s)\, ds \geq \frac{1}{e} + \lambda_k \left(\int_{t_k}^{t_{k+1}} p(s)\, ds - \frac{1}{e} \right) \quad \text{for } t_k < t \leq t_{k+1},\ k = 1, 2, \ldots,$$

for some $\lambda_k \geq 0$, and

$$\liminf_{k\to\infty} \lambda_k = \lambda > 0.$$

We remark that if $\int_{\tau(t)}^{t} p(s)\, ds$ is a nonincreasing function and $\int_{\tau(t)}^{t} p(s)\, ds > \frac{1}{e}$, then $p(t) \in \mathscr{A}_1$, because we may have $\lambda_k = 1$ in (5). However, the monotonicity is not a necessary condition; e.g., in the case $\tau(t) = t - 1$ the function

$$p(t) = \tfrac{1}{e} + (K \sin^2 \pi t / t^\alpha), \qquad K > 0 \text{ and } 0 \leq \alpha \leq 2, \tag{6}$$

belongs to $\mathscr{A}_1$ because $\int_{t-1}^{t} (\sin^2 \pi s / s^\alpha)\, ds$ is a nonincreasing function.

Our main results are

Theorem 1. *Assume that the function $p(t)$ in (1) belongs to $\mathscr{A}_\lambda$ for some $\lambda \in (0, 1]$ and*

$$\sum_{i=1}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)\, ds - \frac{1}{e} \right) = +\infty. \tag{7}$$

Then every solution of (1) oscillates.

In the next theorem we consider the case where the sum in (7) is convergent.

Theorem 2. *Assume that* $p(t) \in \mathscr{A}_\lambda$, *for some* $0 < \lambda \leq 1$ *and either*

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)\,ds - \frac{1}{e} \right) > \frac{2}{e} \tag{8}$$

or

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)\,ds - \frac{1}{e} \right) > \frac{1}{2e}. \tag{9}$$

Then every solution of (1) *oscillates.*

Note. If the function $\int_{\tau(t)}^{t} p(s)\,ds$ is monotone, then the value of λ in conditions (8) and (9) of Theorem 2 is equal to one.

In the following theorem we give a criterion for nonoscillation.

Theorem 3. *Let* $\tau(t) = t - 1$, $p(t) = \frac{1}{e} + a(t)$, *and* $t_0 = 1$ *in* (1); *i.e., it has the form*

$$x'(t) + [\tfrac{1}{e} + a(t)]x(t-1) = 0, \qquad t \geq 1. \tag{1$'$}$$

Assume that

$$a(t) \leq 1/8et^2.$$

Then $(1)'$ *has a soluion* $x(t) \geq \sqrt{t}e^{-t}$.

The proofs of the above theorems and also some lemmas which will be used in these proofs will be given in the next section.

2. Lemmas and proofs

The first two lemmas have origin in [3] (see also [2]).

Lemma 1. *Assume that* $x(t)$ *is a positive solution of* (1) *on* $[t_{k-2}, t_{k+1}]$ *for some* $k \geq 2$. *Let* N *be defined by*

$$N = \min_{t_k \leq t \leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Then $N < (2e)^2$.

Proof. Let L be the integral

$$L = \int_{t_k}^{t_{k+1}} p(s)\,ds \geq \frac{1}{e}.$$

By Lemma 3 in [2], we obtain $N < ((1 + \sqrt{1-L})/L)^2$. Since the right-hand side is a decreasing function of L, we get

$$N < ((1 + \sqrt{1 - (1/e)})/L)^2 < (2e)^2.$$

Lemma 2. *Assume that* $x(t)$ *is a positive solution of* (1) *on* $[t_{k-3}, t_{k+1}]$ *for some* $k \geq 3$ *and* $p(t) \in \mathscr{A}_\lambda$. *Let* M, N *be defined by*

$$M = \min_{t_{k-1} \leq t \leq t_k} \frac{x(\tau(t))}{x(t)}, \qquad N = \min_{t_k \leq t \leq t_{k-1}} \frac{x(\tau(t))}{x(t)}.$$

Then

$$M > 1 \quad \text{and} \quad N \geq \exp\left(M\left[\frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)\,ds - \frac{1}{e}\right)\right]\right) \geq M.$$

Proof. Following the lines of the proof of Lemma 1 in [2], we have $\min\{M, N\} = M$, and by (5) for $t_k \leq t \leq t_{k+1}$

$$\frac{x(\tau(t))}{x(t)} \geq \exp\left(M\int_{\tau(t)}^{t} p(s)\,ds\right) \geq \exp\left(M\left[\frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)\,ds - \frac{1}{e}\right)\right]\right),$$

which implies the inequality concerning N. On the other hand the solution $x(t)$ is a strictly decreasing function on $[t_{k-2}, t_{k+1}]$. Hence $x(\tau(t))/x(t) > 1$ on $[t_{k-1}, t_k]$, and therefore $M > 1$. The proof of the lemma is complete.

The next lemma deals with some properties of the following sequence.

Let the sequence $\{r_i\}_{i=0}^{\infty}$ be defined by the recurrence relation

$$r_0 = 1, \qquad r_{i+1} = e^{r_i/e} \quad \text{for } i = 0, 1, 2, \dots. \tag{10}$$

Lemma 3. *For the sequence* $\{r_i\}_{i=0}^{\infty}$ *in* (10) *the following relations hold:*

(a) $r_i < r_{i+1}$;
(b) $r_i < e$;
(c) $\lim_{i\to\infty} r_i = e$;
(d) $r_i > e - 2e/(i+2)$.

Proof. The first two relations can be proved by induction. As a consequence of (a) and (b) the $\lim_{i\to\infty} r_i = r$ exists and it is finite. Then by (10) we have

$$r = e^{r/e}.$$

It is easy to check that

$$e^{x/e} > x \quad \text{for } x \neq e. \tag{11}$$

This inequality implies that the limit r equals e.

Now we give the proof of (d). For $i = 0$ and $i = 1$ it is immediate. For $i \geq 1$ the proof goes by induction, so we have

$$r_{i+1} = e^{r_i/e} > e^{1-2/(i+2)},$$

and it is sufficient to show

$$e^{1-2/(i+2)} > e - \frac{2e}{i+3},$$

or

$$f(x) = e^{-2/x} + \frac{2}{x+1} > 1 \quad \text{for } x = i+2.$$

Since

$$f'(x) = \frac{2}{x^2}\left(e^{-1/x} + \frac{x}{x+1}\right)\left(e^{-1/x} - \frac{x}{x+1}\right)$$

and

$$e^{1/x} > 1 + \frac{1}{x} = \frac{x+1}{x},$$

we have $f'(x) < 0$ and $f(x) > \lim_{x\to\infty} f(x) = 1$, which was to be shown.

The proof of the lemma is complete.

Proof of Theorem 1. Suppose the contrary. Then we may assume, without loss of generality, that there exists a solution $x(t)$ such that $x(t) > 0$ for $t \geq t_{k-3}$ for some $k \geq 3$. Let the sequence $\{N_i\}_{i=0}^{\infty}$ be defined by

$$N_i = \min_{t_{k+i-1} \leq t \leq t_{k+i}} \frac{x(\tau(t))}{x(t)}. \tag{12}$$

By Lemma 2 we have $N_0 > 1$ and

$$N_{i+1} \geq \exp\left(\frac{N_i}{e}\right) \exp\left(N_i \lambda_{k+i} \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right)\right) \geq N_i; \tag{13}$$

therefore the sequence $\{N_i\}_{i=0}^{\infty}$ is nondecreasing. On the other hand, by Lemma 1, it is bounded. Consequently the sequence converges. Let

$$\lim_{i \to \infty} N_i = N.$$

Then (13) implies

$$N \geq \exp(N/e).$$

Hence by (11) we have $N = e$ and

$$1 < N_0 < N_1 < \cdots < e. \tag{14}$$

From (13), in view of (11), we obtain

$$N_{i+1} \geq N_i \left(1 + N_i \lambda_{k+i} \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right)\right).$$

Thus

$$N_{i+1} - N_i > N_i^2 \lambda_{k+i} \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right). \tag{15}$$

From the definition of $\mathscr{A}_\lambda$ we know that $\lambda = \liminf_{k \to \infty} \lambda_k > 0$, so for any sufficiently small $\varepsilon > 0$ there exists a value κ_ε such that $\lambda_{k+i} > \lambda - \varepsilon$ for $k + i > \kappa_\varepsilon$. Thus, for such i's from (15) and (14), we have

$$\begin{aligned}
N_{i+1} - N_i &> N_i^2 (\lambda - \varepsilon) \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right), \\
N_{i+2} - N_{i+1} &> N_{i+1}^2 (\lambda - \varepsilon) \left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)\,ds - \frac{1}{e}\right) \\
&\geq N_i^2 (\lambda - \varepsilon) \left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)\,ds - \frac{1}{e}\right). \\
&\vdots
\end{aligned}$$

Summing up the inequalities above, we obtain

$$e - N_i > N_i^2 (\lambda - \varepsilon) \sum_{j=i}^{\infty} \left(\int_{t_{k+j}}^{t_{k+j+1}} p(s)\,ds - \frac{1}{e}\right) \quad \text{for } k + i \geq \kappa_\varepsilon. \tag{16}$$

The last inequality contradicts assumption (7). The proof is complete.

Proof of Theorem 2. Suppose the contrary. Then, as in the proof of Theorem 1, we have the sequence $\{N_i\}_{i=0}^{\infty}$ such that inequalities (13)–(16) hold. In particular, from (13) we have

$$N_{i+1} \geq \exp(N_i/e).$$

Comparing the last inequality with (10), we obtain by induction

$$N_0 > r_0 = 1\,, \qquad N_i > r_i \quad \text{for } i = 1\,, 2\,, \ldots\,.$$

Then by Lemma 3(d) we have

$$e - N_i < e - r_i < 2e/(i+2). \tag{17}$$

Multiplying (16) by $k+i$ we obtain

$$(k+i)\frac{2e}{i+2} > N_i^2(\lambda - \varepsilon)(k+i)\sum_{j=k+i}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)\,ds - \frac{1}{e}\right) \quad \text{for } k+i \geq \kappa_\varepsilon.$$

Taking the limit as $i \to \infty$ we get

$$2e \geq e^2\lambda \limsup_{k\to\infty} k\sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)\,ds - \frac{1}{e}\right)\,,$$

which contradicts (8).

Now let A be defined by

$$A = \liminf_{k\to\infty} k\sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)\,ds - \frac{1}{e}\right).$$

If $A = \infty$, then, by (8), every solution oscillates. Therefore we consider the case $0 < A < \infty$. So for any sufficiently small $\varepsilon > 0$ there exists a value $\hat{\kappa}_\varepsilon$ such that for $\hat{\lambda} = \lambda - \varepsilon > 0$ and $\hat{A} = A - \varepsilon > 0$

$$\lambda_k > \hat{\lambda} \quad \text{and} \quad \sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)\,ds - \frac{1}{e}\right) > \frac{\hat{A}}{k} \quad \text{for } k \geq \hat{\kappa}_\varepsilon. \tag{18}$$

If we use the inequality

$$\exp\frac{x}{e} > x + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1 - \frac{x}{e}\right)^2 \quad \text{for } \xi < x < e$$

in (13) we obtain for $N_i > \xi$ and $k+i > \hat{\kappa}_\varepsilon$

$$\begin{aligned} N_{i+1} &\geq \exp\left(\frac{N_i}{e}\right)\exp\left(N_i\hat{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right)\right) \\ &> \left[N_i + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1 - \frac{N_i}{e}\right)^2\right]\left(1 + N_i\hat{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right)\right). \end{aligned}$$

Consequently

$$N_{i+1} - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1 - \frac{N_i}{e}\right)^2 + \xi^2\hat{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)\,ds - \frac{1}{e}\right)$$

$$\vdots$$

and summing up,

$$e - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(1-\frac{N_i}{e}\right)^2 + \xi^2\hat{\lambda}\sum_{j=k+i}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)\,ds - \frac{1}{e}\right)$$

or

$$e - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(1-\frac{N_i}{e}\right)^2 + \frac{\xi^2\hat{\lambda}\hat{A}}{k+i}. \tag{19}$$

In particular the last inequality yields

$$e - N_i > U_0/(k+i), \qquad U_0 = \xi^2\hat{\lambda}\hat{A}.$$

By iteration we can improve this inequality to

$$e - N_i > \frac{U_n}{k+i}, \qquad n = 0, 1, 2, \ldots. \tag{20}$$

Namely by (19) we have

$$\begin{aligned} e - N_i &> \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(\frac{U_n}{e(k+j)}\right)^2 + \frac{\xi^2\hat{\lambda}\hat{A}}{k+i} \\ &> \frac{U_n^2}{2e^2}\exp\left(\frac{\xi}{e}\right)\frac{1}{k+i} + \frac{\xi^2\hat{\lambda}\hat{A}}{k+i} = \frac{U_{n+1}}{k+i}, \end{aligned}$$

where

$$U_{n+1} = \frac{U_n^2}{2e^2}\exp\left(\frac{\xi}{e}\right) + \xi^2\hat{\lambda}\hat{A}, \qquad n = 0, 1, 2, \ldots. \tag{21}$$

From this it is clear that the sequence $\{U_n\}_{n=0}^{\infty}$ is increasing. Moreover, comparing inequalities (17) and (20) we see that $U_n \le 2e$. Therefore the sequence has a limit, say U, which satisfies the equation

$$U = \frac{U^2}{2e^2}\exp\left(\frac{\xi}{e}\right) + \xi^2\hat{\lambda}\hat{A}.$$

This is a quadratic equation with real roots and therefore the descriminant is not negative; i.e.,

$$1 - 2e^{\xi/e-2}\xi^2\hat{\lambda}\hat{A} \ge 0.$$

Let $\varepsilon \to 0$ and $\xi \to e$. Then the last inequality becomes

$$1 - 2e\lambda A \ge 0,$$

which contradicts (9).

The proof of the theorem is complete.

Proof of Theorem 3. The proof is based on known comparison theorems (see Myshkis [6] or Elbert [1]). Let the functions $A(t)$, $B(t)$, $C(t)$ be defined as

$$\begin{aligned} A(t) &= \frac{1}{e} + a(t), \\ B(t) &= \frac{1}{e} + \frac{1}{8et^2}, \\ C(t) &= \frac{1}{e}\frac{1-\frac{1}{2t}}{\sqrt{1-\frac{1}{t}}}, \qquad t > 1. \end{aligned}$$

By the assumption we have $A(t) \leq B(t)$. We are going to show that the inequality $B(t) < C(t)$ also holds. Namely, for $\theta = \frac{1}{2t} \in (0, \frac{1}{2})$, we have

$$C(t) - B(t) = \frac{\theta^3(\frac{1}{2}\theta^2 - \frac{1}{4}\theta + 2)}{e\sqrt{1-2\theta}[1 - \theta + (1 + \frac{1}{2}\theta^2)\sqrt{1-2\theta}} > 0.$$

Now we will compare the differential equations

$$x'(t) + A(t)x(t-1) = 0,$$
$$z'(t) + B(t)z(t-1) = 0,$$
$$u'(t) + C(t)u(t-1) = 0.$$

Let us observe that the function $u(t) = \sqrt{t}e^{-t}$ is a solution of the last differential equation. Let the initial function $\varphi(t)$ be the function $\sqrt{t}e^{-t}$ on [0, 1], and let $x(t)$ and $z(t)$ be the solutions of the first and the second differential equations respectively, associated with this initial function $\varphi(t)$. Then by the comparison theorems mentioned above we have

$$x(t) \geq z(t) > u(t) = \sqrt{t}e^{-t} \quad \text{for } t > 1.$$

which was to be shown.

Remark 1. For $(1)'$ we have $t_k = k + 1$ and

$$\limsup_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)\,ds - \frac{1}{e} \right) = \limsup_{k\to\infty} k \int_k^{\infty} a(t)\,dt \leq \frac{1}{8e}.$$

Now the question arises naturally whether or not the bounds in conditions (8) and (9) of Theorem 2 can be replaced by smaller ones.

Remark 2. It is to be emphasized that in Theorem 3 we require neither

$$p(t) \geq 0 \quad \text{nor} \quad \int_{\tau(t)}^{t} p(s)\,ds \geq \frac{1}{e}.$$

Remark 3. Applying Theorems 1, 2 we see that, under (6), (1) oscillates for any $K > 0$ if $0 \leq \alpha < 2$ and for $K > \frac{1}{e}$ if $\alpha = 2$. On the other hand it has a nonoscillatory solution for $K < \frac{1}{8e}$ if $\alpha = 2$.

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