A CONGRUENCE FOR PRIMES

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ABSTRACT. With the help of the Pell sequence we obtain the following new congruence for odd primes:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

1. INTRODUCTION

Congruences for primes are of great interest. Some examples are

$$a^p \equiv a \pmod{p} \quad (\text{Fermat's little theorem}),$$

$$(p-1)! \equiv -1 \pmod{p} \quad (\text{Wilson's theorem}),$$

$$\binom{2p-1}{p} \equiv 1 \pmod{p^3} \text{ for primes } p > 3 \quad (\text{Wolstenholme's theorem}).$$

In this paper we shall establish the following congruence for odd primes:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}$$

where $[\cdot]$ is the greatest integer function.

2. SOME PREPARATIONS

Let

$$\begin{bmatrix} n \\ r \end{bmatrix} = \sum_{\substack{k=0 \\ k \equiv r \pmod{8}}}^n \binom{n}{k}.$$

Using

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{and} \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

one can easily prove that

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}, \quad \begin{bmatrix} n+1 \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} + \begin{bmatrix} n \\ r-1 \end{bmatrix}.$$

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So we have

$$\begin{aligned}\begin{bmatrix} n+2 \\ 2r \end{bmatrix} &= \begin{bmatrix} n+1 \\ 2r \end{bmatrix} + \begin{bmatrix} n+1 \\ 2r-1 \end{bmatrix} = \begin{bmatrix} n \\ 2r \end{bmatrix} + \begin{bmatrix} n \\ 2r-1 \end{bmatrix} + \begin{bmatrix} n \\ 2r-1 \end{bmatrix} + \begin{bmatrix} n \\ 2r-2 \end{bmatrix} \\ &= \begin{bmatrix} n \\ 2r \end{bmatrix} + 2 \begin{bmatrix} n \\ 2((n+1)/2-r) \end{bmatrix} + \begin{bmatrix} n \\ 2(r-1) \end{bmatrix}.\end{aligned}$$

By induction on n , one can prove the following result (due to Zhi-Hong Sun [1]) involving the Pell sequence $\{P_n\}$ ($P_0 = 0$, $P_1 = 1$, $P_{k+1} = 2P_k + P_{k-1}$) and its companion $\{Q_n\}$ ($Q_0 = 2$, $Q_1 = 2$, $Q_{k+1} = 2Q_k + Q_{k-1}$).

Lemma 1. *Let $n > 0$ be odd. We have*

(i) *if $n \equiv 1 \pmod{8}$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} + (-1)^r 2^{(n-5)/2} + (-1)^{[r/2] + (n-1)/8} 2^{(n-5)/4} P_{(n+(-1)^r)/2};$$

(ii) *if $n \equiv 3 \pmod{8}$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} - (-1)^r 2^{(n-5)/2} + (-1)^{[r/2] + (n-3)/8} 2^{(n-11)/4} Q_{(n-(-1)^r)/2};$$

(iii) *if $n \equiv 5 \pmod{8}$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} - (-1)^r 2^{(n-5)/2} + (-1)^{[(r+1)/2] + (n+3)/8} 2^{(n-5)/4} P_{(n-(-1)^r)/2};$$

(iv) *if $n \equiv 7 \pmod{8}$, then*

$$\begin{bmatrix} n \\ 2r \end{bmatrix} = 2^{n-3} + (-1)^r 2^{(n-5)/2} + (-1)^{[(r+1)/2] + (n+1)/8} 2^{(n-11)/4} Q_{(n+(-1)^r)/2}.$$

From Lemma 1 it follows that for odd $n > 0$

$$\begin{aligned}n \sum_{k=1}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1} &= 4 \sum_{\substack{k=1 \\ 4|k}}^n \frac{n}{k} \binom{n-1}{k-1} \\ &= 4 \sum_{\substack{k=0 \\ 4|k}}^n \binom{n}{k} - 4 = 4 \left(\begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 4 \end{bmatrix} \right) - 4 \\ &= 4 \times (2^{n-2} + (-1)^{(n^2-1)/8} 2^{(n-3)/2}) - 4 \\ &= 2(2^{n-1} - 1) + 2(-1)^{(n^2-1)/8} \\ &\quad \times (2^{(n-1)/2} - (-1)^{(n^2-1)/8}), \\ n \sum_{k=1}^{[(n+1)/4]} \frac{1}{2k-1} \binom{n-1}{4k-3} &= 2 \sum_{\substack{k=0 \\ 4|k+2}}^n \frac{n}{k} \binom{n-1}{k-1} = 2 \left(\begin{bmatrix} n \\ 2 \end{bmatrix} + \begin{bmatrix} n \\ 6 \end{bmatrix} \right) \\ &= 2(2^{n-2} - (-1)^{(n^2-1)/8} 2^{(n-3)/2}) \\ &= 2^{n-1} - 1 - (-1)^{(n^2-1)/8} (2^{(n-1)/2} - (-1)^{(n^2-1)/8}).\end{aligned}$$

So we have

Lemma 2. *Let $n > 0$ be odd. Then*

$$F_n + (-1)^{(n^2-1)/8} E_n = \frac{1}{2} \sum_{k=1}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1},$$

$$F_n - (-1)^{(n^2-1)/8} E_n = \sum_{k=1}^{[(n+1)/4]} \frac{1}{2k-1} \binom{n-1}{4k-3}$$

where

$$F_n = \frac{2^{n-1} - 1}{n} \quad \text{and} \quad E_n = (-1)^{(n^2-1)/8} \frac{2^{(n-1)/2} - (-1)^{(n^2-1)/8}}{n}.$$

For the sequences $\{P_n\}$ and $\{Q_n\}$, it is well known that

$$P_n = \frac{1}{2\sqrt{2}} [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n], \quad Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

From this one can easily derive that

$$P_{n+1}^2 + P_n^2 = P_{2n+1} = \frac{1}{8} (Q_{n+1}^2 + Q_n^2).$$

This property will be used below.

Lemma 3. *Suppose that $n > 0$ is odd. Then*

$$2^{(n-1)/2} P_n = 1 + \frac{n}{2} \sum_{k=1}^{[(n-1)/4]} \frac{(-1)^k}{k} \binom{n-1}{4k-1}$$

$$+ \frac{n^2}{4} \left[\left(\sum_{k=1}^{[(n-1)/4]} \frac{(-1)^k}{2k} \binom{n-1}{4k-1} \right)^2 + \left(\sum_{k=1}^{[(n+1)/4]} \frac{(-1)^{k-1}}{2k-1} \binom{n-1}{4k-3} \right)^2 \right].$$

Proof. By Lemma 1 we have

$$\left(\left[\begin{matrix} n \\ 2r \end{matrix} \right] - 2^{n-3} - (-1)^{r+(n^2-1)/8} 2^{(n-5)/2} \right)^2$$

$$+ \left(\left[\begin{matrix} n \\ 2(r+1) \end{matrix} \right] - 2^{n-3} - (-1)^{r+1+(n^2-1)/8} 2^{(n-5)/2} \right)^2$$

$$= \begin{cases} 2^{(n-5)/2} (P_{(n+1)/2}^2 + P_{(n-1)/2}^2) = 2^{(n-5)/2} P_n & \text{if } n \equiv 1 \pmod{4}, \\ 2^{(n-11)/2} (Q_{(n+1)/2}^2 + Q_{(n-1)/2}^2) = 2^{(n-5)/2} P_n & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Putting $r = 0$ and noticing that

$$\left[\begin{matrix} n \\ 0 \end{matrix} \right] = 1 + \sum_{\substack{k=1 \\ 8|k}}^{n-1} \binom{n}{k} = 1 + nC_n, \quad \left[\begin{matrix} n \\ 2 \end{matrix} \right] = \sum_{\substack{k=1 \\ k \equiv 2 \pmod{8}}}^{n-1} \binom{n}{k} = nD_n$$

where

$$C_n = \sum_{k=1}^{[(n-1)/8]} \frac{1}{8k} \binom{n-1}{8k-1} \quad \text{and} \quad D_n = \sum_{k=1}^{[(n+5)/8]} \frac{1}{8k-6} \binom{n-1}{8k-7},$$

we then get

$$\begin{aligned} 2^{(n+3)/2} P_n &= \left(4 \begin{bmatrix} n \\ 0 \end{bmatrix} - 2^{n-1} - (-1)^{(n^2-1)/8} 2^{(n-1)/2} \right)^2 \\ &\quad + \left(4 \begin{bmatrix} n \\ 2 \end{bmatrix} - 2^{n-1} + (-1)^{(n^2-1)/8} 2^{(n-1)/2} \right)^2 \\ &= [4 + 4nC_n - (1 + nF_n) - (-1)^{(n^2-1)/8} ((-1)^{(n^2-1)/8} + nE_n)]^2 \\ &\quad + [4nD_n - (1 + nF_n) + (-1)^{(n^2-1)/8} ((-1)^{(n^2-1)/8} + nE_n)]^2 \\ &= [2 + n(4C_n - F_n - (-1)^{(n^2-1)/8} E_n)]^2 \\ &\quad + n^2[4D_n - F_n + (-1)^{(n^2-1)/8} E_n]^2 \\ &= 4 + 4n(4C_n - F_n - (-1)^{(n^2-1)/8} E_n) \\ &\quad + n^2[(4C_n - F_n - (-1)^{(n^2-1)/8} E_n)^2 \\ &\quad + (4D_n - F_n + (-1)^{(n^2-1)/8} E_n)^2]. \end{aligned}$$

Using Lemma 2 we find that

$$\begin{aligned} 4C_n - F_n - (-1)^{(n^2-1)/8} E_n &= \sum_{k=1}^{[(n-1)/8]} \frac{1}{2k} \binom{n-1}{8k-1} - \frac{1}{2} \sum_{k=1}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1} \\ &= \sum_{\substack{k=1 \\ 2|k}}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1} - \frac{1}{2} \left[\sum_{\substack{k=1 \\ 2|k}}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1} + \sum_{\substack{k=1 \\ 2 \nmid k}}^{[(n-1)/4]} \frac{1}{k} \binom{n-1}{4k-1} \right] \\ &= \frac{1}{2} \sum_{k=1}^{[(n-1)/4]} \frac{(-1)^k}{k} \binom{n-1}{4k-1}, \\ 4D_n - F_n + (-1)^{(n^2-1)/8} E_n &= 4 \sum_{k=1}^{[(n+5)/8]} \frac{1}{8k-6} \binom{n-1}{8k-7} - \sum_{k=1}^{[(n+1)/4]} \frac{1}{2k-1} \binom{n-1}{4k-3} \\ &= 2 \sum_{\substack{k=1 \\ 2|k+1}}^{[(n+1)/4]} \frac{1}{2k-1} \binom{n-1}{4k-3} - \sum_{k=1}^{[(n+1)/4]} \frac{1}{2k-1} \binom{n-1}{4k-3} \\ &= \sum_{k=1}^{[(n+1)/4]} \frac{(-1)^{k-1}}{2k-1} \binom{n-1}{4k-3}. \end{aligned}$$

This concludes the proof. $\square$

Lemma 4. Let $n > 0$ be odd. Then

$$P_n = 2^{(n-1)/2} + 2^{(n-3)/2} n \sum_{k=1}^{(n-1)/2} \frac{1}{k \cdot 2^k} \binom{n-1}{2k-1}.$$

Proof.

$$\begin{aligned} P_n - 2^{(n-1)/2} &= \frac{1}{2\sqrt{2}} [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n] - 2^{(n-1)/2} \\ &= \frac{1}{2\sqrt{2}} \times 2 \sum_{\substack{k=0 \\ 2 \nmid k}}^n \binom{n}{n-k} (\sqrt{2})^k - (\sqrt{2})^{n-1} \\ &= \sum_{i=1}^{(n-1)/2} \binom{n}{n+1-2i} 2^{i-1} \\ &= \frac{1}{2} \sum_{i=1}^{(n-1)/2} \frac{n}{(n+1)/2-i} \binom{n-1}{2((n+1)/2-i)-1} 2^{(n-1)/2-((n+1)/2-i)} \\ &= \frac{n}{2} \sum_{k=1}^{(n-1)/2} \frac{1}{k} \binom{n-1}{2k-1} 2^{(n-1)/2-k} \\ &= 2^{(n-3)/2} n \sum_{k=1}^{(n-1)/2} \frac{1}{k \cdot 2^k} \binom{n-1}{2k-1}. \quad \square \end{aligned}$$

3. THE MAIN RESULT

Theorem. Let p be an odd prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

Proof. From Lemmas 3 and 4 it follows that

$$\begin{aligned} 2^{p-2} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \binom{p-1}{2k-1} &= \frac{2^{(p-1)/2} P_p - 2^{p-1}}{p} \\ &= -\frac{2^{p-1}-1}{p} + \frac{1}{2} \sum_{k=1}^{[(p-1)/4]} \frac{(-1)^k}{k} \binom{p-1}{4k-1} \\ &\quad + \frac{p}{4} \left[\left(\sum_{k=1}^{[(p-1)/4]} \frac{(-1)^k}{2k} \binom{p-1}{4k-1} \right)^2 \right. \\ &\quad \left. + \left(\sum_{k=1}^{[(p+1)/4]} \frac{(-1)^{k-1}}{2k-1} \binom{p-1}{4k-3} \right)^2 \right]. \end{aligned}$$

Since p is an odd prime, we have

$$2^{p-1} \equiv 1 \pmod{p}, \quad \binom{p-1}{n} = \frac{(p-1)(p-2) \cdots (p-n)}{1 \times 2 \times \cdots \times n} \equiv (-1)^n \pmod{p},$$

and hence

$$\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} (-1)^{2k-1} \equiv -\frac{1}{2} \times \frac{2^p - 2}{p} + \frac{1}{2} \sum_{k=1}^{[(p-1)/4]} \frac{(-1)^k}{k} (-1)^{4k-1} \pmod{p}.$$

Thus

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} &\equiv \frac{\sum_{k=1}^{p-1} \binom{p}{k}}{p} + \sum_{k=1}^{[(p-1)/4]} \frac{(-1)^k}{k} \equiv \sum_{k=1}^{p-1} \frac{1}{k} \binom{p-1}{k-1} \\ &\quad + \sum_{k=1}^{[(p-1)/4]} \frac{(-1)^{p-k}}{p-k} \pmod{p} \\ &\equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{k=p-[(p-1)/4]}^{p-1} \frac{(-1)^k}{k} \\ &\equiv \sum_{k=1}^{p-1-[(p-1)/4]} \frac{(-1)^{k-1}}{k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}. \end{aligned}$$

This completes the proof. $\square$

Remark. The congruence we proved was first conjectured by the author's brother Zhi-Hong Sun in the summer of 1988. Although he failed to prove the conjecture, he was able to show the congruence

$$P_p - 2^{(p-1)/2} \equiv -2^{(p-3)/2} p \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \pmod{p^2}$$

and confirmed the conjecture in the case $p \equiv 1 \pmod{8}$.

To end this paper we mention another beautiful congruence (for odd primes)

$$\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv \sum_{k=1}^{[(p-1)/6]} \frac{(-1)^k}{k} \pmod{p}.$$

It will be published in [2]. For related work the reader can consult [3] and [4].

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