# AN ESTIMATION OF SINGULAR VALUES OF CONVOLUTION OPERATORS 

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(Communicated by Palle E. T. Jorgensen)


#### Abstract

In this paper we determine the asymptotic order of singular values of convolution operators $\int_{0}^{x} k(x-y) \cdot d y$, where $k(x)=x^{\alpha-1} L(1 / x) \quad(0<$ $\alpha<1 / 2$ ) and $L$ is a slowly varying function from some class.


## 1. Introduction

Let $\mathscr{H}$ be a separable Hilbert space over $\mathbb{C}$ and $A$ be a compact operator. The singular values of $A\left(s_{n}(A)\right)$ are the eigenvalues of the operator $\left(A^{*} A\right)^{1 / 2}$ (or $\left(A A^{*}\right)^{1 / 2}$ ).
V. Faber and G. M. Wing [3, 4] have found an upper bound on the singular values of fractional integral operators and of some other similar operators.

In [2] an exact asymptotic of the singular values of the fractional integral operator $I^{\alpha}=\frac{1}{\Gamma(\alpha)} \int_{0}^{x}(x-y)^{\alpha-1} \cdot d y$ is found. In this paper we find the asymptotic order of the singular values of the operator $\int_{0}^{x} k(x-y) \cdot d y$ acting on $\mathscr{H}=L^{2}(0,1)$ whose kernel has power singularity and singularity arising from a slowly varying function $L$ in the point $x=0$. In what follows for given sequences $\left\{a_{n}\right\},\left\{b_{n}\right\} \quad\left(a_{n}>0, b_{n}>0\right)$ we write $a_{n} \asymp b_{n}$ if there exist constants $c_{1}, c_{2}>0$ such that $c_{1} \leq a_{n} / b_{n} \leq c_{2}$ for all $n \in \mathbb{N}$. By $\int_{a}^{b} m(x, y) \cdot d y$ we denote the integral operator on $L^{2}(a, b)$ with the kernel $m(x, y)$.

## 2. Main result

Let $L \in C^{1}[1, \infty)$ be a nondecreasing function on $[1, \infty)$, let

$$
\lim _{x \rightarrow+\infty} x L^{\prime}(x) / L(x)=0
$$

and let $x \mapsto x L^{\prime}(x) / L(x)$ be a nonincreasing function for $x$ large enough. Define the operator $A: L^{2}(0,1) \rightarrow L^{2}(0,1)$ by

$$
A f(x)=\int_{0}^{x} k(x-y) f(y) d y
$$

Received by the editors July 2, 1993.
1991 Mathematics Subject Classification. Primary 47A70.
Key words and phrases. Singular values, convolution operators, slowly varying function.
where

$$
k(x)=x^{\alpha-1} L\left(\frac{1}{x}\right) \quad(\alpha>0)
$$

Theorem 1. If $0<\alpha<1 / 2$, then $s_{n}(A) \asymp L(n) / n^{\alpha}$.
Proof. Case A: $L$ is not a bounded function. Then $\lim _{x \rightarrow+\infty} L(x)=+\infty$.
Observe that if we smoothly extend $L$ from $[1, \infty)$ to $[0, \infty)$ the new operator $A$ has the same singular values as the old one (because it acts on $\left.\mathscr{H}=L^{2}(0,1)\right)$. Because of that we can assume $L \in C^{1}[0, \infty)$ and that $L$ is a linear function on $[0,1]$. Without loss of generality we can assume that $L>0$ on $[0, \infty)$ (because $s_{n}\left(I^{\alpha}\right) \asymp 1 / n^{\alpha}$ [2]). Let $a>1$ be a fixed number and let

$$
L_{a}(x)= \begin{cases}L(x), & x \geq a \\ L^{\prime}(a) x+L(a)-a L^{\prime}(a), & 0 \leq x \leq a\end{cases}
$$

Let $B$ and $B_{a}$ be linear operators on $L^{2}(0,1)$ defined by

$$
\begin{aligned}
B f(x) & =\int_{0}^{1}|x-y|^{\alpha-1} L\left(\frac{1}{|x-y|}\right) f(y) d y \\
B_{a} f(x) & =\int_{0}^{1}|x-y|^{\alpha-1} L_{a}\left(\frac{1}{|x-y|}\right) f(y) d y
\end{aligned}
$$

Before the proof of Theorem 1 we give the following lemma.
Lemma 1. If $0<\alpha<1 / 2$ and $a$ is large enough, then.

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{s_{n}(B)}{s_{n}\left(B_{a}\right)}=1 \tag{1}
\end{equation*}
$$

Proof. Let $P: L^{2}(0,1) \rightarrow L^{2}(0,1)$ be a linear operator defined by $\operatorname{Pf}(x)=$ $\chi_{[0,1 / a]}(x) f(x)$ and $Q=I-P$. (Here $\chi_{[a, b]}$ is the characteristic function of $[a, b]$.) Then

$$
\begin{aligned}
B_{a} & =(P+Q) B_{a}(P+Q) \\
& =P B_{a} P+Q B_{a} P+P B_{a} Q+Q B_{a} Q
\end{aligned}
$$

and

$$
\begin{aligned}
B & =(P+Q) B(P+Q) \\
& =P B P+Q B P+P B Q+Q B Q .
\end{aligned}
$$

Since $L_{a}(1 / x)=L(1 / x)$ for $0<x \leq 1 / a$, we obtain $P B_{a} P=P B P$ and

$$
\begin{equation*}
B=B_{a}+Q\left(B-B_{a}\right) P+P\left(B-B_{a}\right) Q+Q\left(B-B_{a}\right) Q \tag{2}
\end{equation*}
$$

From the definition of $L_{a}$ it follows that $Q\left(B-B_{a}\right) P$ and $Q\left(B-B_{a}\right) Q$ are Hilbert Schmidt operators and hence

$$
\begin{align*}
& s_{n}\left(Q\left(B-B_{a}\right) P+P\left(B-B_{a}\right) Q+Q\left(B-B_{a}\right) Q\right)  \tag{3}\\
& \quad=\sigma\left(n^{-1 / 2}\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right) \quad\left(0<\alpha<\frac{1}{2}\right)
\end{align*}
$$

If we show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{n^{\alpha}}{L(n)} s_{n}\left(B_{a}\right)=c_{0} \neq 0 \tag{4}
\end{equation*}
$$

then from (2), (3), (4), and the Ky Fan Theorem [5] follows (1). Now we prove (4) (with $c_{0}=\pi^{-\alpha} \Gamma(\alpha) \cos (\alpha \pi / 2)$ ) if $0<\alpha<1 / 2$ and $a$ is large enough.

Consider the operator $B_{a}^{\prime}: L^{2}(-1,1) \rightarrow L^{2}(-1,1)$ defined by

$$
B_{a}^{\prime} f(x)=\int_{-1}^{1} k_{a}(|x-y|) f(y) d y
$$

where

$$
k_{a}(t)=t^{\alpha-1} L_{a}\left(\frac{1}{t}\right) \quad(t>0)
$$

Let

$$
K_{a}(\xi)=\int_{\mathbb{R}} e^{i t \xi} k_{a}(|t|) d t
$$

and

$$
H_{a}(x, y)=\sum_{n=-\infty}^{\infty}\left(k_{a}(|x-y+4 n|)-k_{a}(|x+y+4 n+2|)\right)
$$

By direct computation we conclude that

$$
\begin{equation*}
\int_{-1}^{1} H_{a}(x, y) \varphi_{n}(y) d y=K_{a}\left(\frac{n \pi}{2}\right) \varphi_{n}(x) \tag{5}
\end{equation*}
$$

where

$$
\varphi_{n}(x)=\sin \frac{n \pi(1+x)}{2}, \quad n \in \mathbb{N} .
$$

(The system $\left\{\varphi_{n}\right\}_{n=1}^{\infty}$ is an orthonormal basis of $L^{2}(-1,1)$.) We shall demonstrate that the following conditions are satisfied:
$1^{\circ} . K_{a}(\xi) \sim$ const $\cdot L(\xi) / \xi^{\alpha}$ when $\xi \rightarrow+\infty$.
$2^{\circ}$. If $a$ is large enough, the function $K_{a}$ is decreasing if $\xi$ is large enough.
$3^{\circ}$. The operator $D: L^{2}(0,2) \rightarrow L^{2}(0,2)$ defined by

$$
D f(x)=\int_{0}^{2} k_{a}(x+y) f(y) d y
$$

has the property $s_{n}(D)=\sigma\left(L(n) / n^{\alpha}\right) \quad(0<\alpha<1 / 2)$.
$4^{\circ}$. The function $\sum_{n \neq 0 ; n \neq-1}\left(k_{a}(|x-y+4 n|)-k_{a}(|x+y+4 n+2|)\right)$ is bounded on $[-1,1]^{2}$.

The property $4^{\circ}$ is the consequence of the linearity of $L_{a}$ on $[0, a]$. Simple computation yields

$$
\int_{0}^{2} \int_{0}^{2}\left|k_{a}(x+y)\right|^{2} d x d y<\infty \quad(\text { for every } \alpha>0)
$$

and hence

$$
s_{n}(D)=\sigma\left(n^{-1 / 2}\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right) \quad\left(0<\alpha<\frac{1}{2}\right)
$$

By a substitution we obtain

$$
K_{a}(\xi)=2 \xi^{-\alpha} \int_{0}^{\infty} t^{-\alpha-1} \cos \frac{1}{t} \cdot L_{a}(\xi t) d t
$$

We shall prove now

$$
\begin{equation*}
Q(\xi)=\int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} L_{a}(\xi x) d x \sim L(\xi) \int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} d x \quad(\xi \rightarrow+\infty) \tag{6}
\end{equation*}
$$

Let

$$
Q_{1}(\xi)=\int_{1 / \pi}^{+\infty} x^{-\alpha-1} \cos \frac{1}{x} L_{a}(\xi x) d x
$$

and

$$
Q_{2}(\xi)=\int_{0}^{1 / \pi} x^{-\alpha-1} \cos \frac{1}{x} L_{a}(\xi x) d x
$$

Then by Theorem 2.6, p. 63 in [6] we get

$$
\begin{align*}
Q_{1}(\xi) & =L_{a}(\xi)\left(\int_{1 / \pi}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} d x+\sigma(1)\right)  \tag{7}\\
& =L(\xi)\left(\int_{1 / \pi}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} d x+\sigma(1)\right)
\end{align*}
$$

By partial integration, we get

$$
Q_{2}(\xi)=\int_{0}^{1 / \pi} \sin \frac{1}{x} \cdot x^{-\alpha} L_{a}(\xi x)\left[1-\alpha+\frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi x)}\right] d x
$$

i.e.,

$$
\begin{aligned}
\frac{Q_{2}(\xi)}{L_{a}(\xi)}= & (1-\alpha) \int_{0}^{1 / \pi} x^{-\alpha} \sin \frac{1}{x} \frac{L_{a}(\xi x)}{L_{a}(\xi)} d x \\
& +\int_{0}^{1 / \pi} x^{-\alpha} \sin \frac{1}{x} \cdot \frac{L_{a}(\xi x)}{L_{a}(\xi)} \cdot \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi x)} d x
\end{aligned}
$$

Since $L_{a}$ is a nondecreasing function and $\lim _{t \rightarrow+\infty} t L_{a}^{\prime}(t) / L_{a}(t)=0$, by the Lebesgue Dominated Convergence Theorem

$$
\frac{Q_{2}(\xi)}{L_{a}(\xi)} \rightarrow(1-\alpha) \int_{0}^{1 / \pi} x^{-\alpha} \sin \frac{1}{x} d x=\int_{0}^{1 / \pi} x^{-\alpha-1} \cos \frac{1}{x} d x
$$

Therefore

$$
\begin{align*}
Q_{2}(\xi) & =L_{a}(\xi)\left(\int_{0}^{1 / \pi} x^{-\alpha-1} \cos \frac{1}{x} d x+\sigma(1)\right)  \tag{8}\\
& =L(\xi)\left(\int_{0}^{1 / \pi} x^{-\alpha-1} \cos \frac{1}{x} d x+\sigma(1)\right)
\end{align*}
$$

Since

$$
\int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} d x=\Gamma(\alpha) \cos \frac{\alpha \pi}{2}
$$

and $Q=Q_{1}+Q_{1}$, from (7) and (8) we obtain (6) and

$$
\begin{equation*}
K_{a}(\xi)=2 \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(\xi)}{\xi^{\alpha}}(1+\sigma(1)), \quad \xi \rightarrow+\infty \tag{9}
\end{equation*}
$$

which proves $1^{\circ}$.

Now we prove property $2^{\circ}$ of $K_{a}$. Since

$$
\begin{aligned}
K_{a}^{\prime}(\xi)= & -2 \alpha \xi^{-\alpha-1} \int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} L_{a}(\xi x) d x \\
& +2 \xi^{-\alpha} \int_{0}^{+\infty} x^{-\alpha} \cos \frac{1}{x} L_{a}^{\prime}(\xi x) d x \\
= & 2 \frac{L_{a}(\xi)}{\xi^{\alpha+1}}\left(-2 \alpha \Gamma(\alpha) \cos \frac{\alpha \pi}{2}+\sigma(1)+\int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi)} d x\right)
\end{aligned}
$$

it suffices to prove that

$$
\begin{equation*}
\left|\int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi)} d x\right|<2 \alpha \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \tag{10}
\end{equation*}
$$

if $a$ and $\xi$ are large enough. Since

$$
\int_{1}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi)} d x \rightarrow 0 \quad(\xi \rightarrow+\infty)
$$

and

$$
\begin{aligned}
& \int_{0}^{a / \xi} x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi)} \\
& \quad=L_{a}^{\prime}(a) \frac{\xi}{L_{a}(\xi)} \int_{0}^{a / \xi} x^{-\alpha} \cos \frac{1}{x} d x \rightarrow 0 \quad(\xi \rightarrow+\infty)
\end{aligned}
$$

inequality (10) will follow from

$$
\begin{equation*}
\left|\int_{a / \xi}^{1} x^{-\alpha-1} \cos \frac{1}{x} \frac{L_{a}(\xi x)}{L_{a}(\xi)} \cdot \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi x)}\right|<2 \alpha \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \tag{11}
\end{equation*}
$$

( $a$ large enough and $\xi>a$ ). Applying twice the Bonet Mean Value Theorem (the functions $x \mapsto L_{a}(x)$ and $x \mapsto x \cdot L_{a}^{\prime}(x) / L_{a}(x)$ are nondecreasing and decreasing on $[0, \infty)$ and $\left[x_{0}, \infty\right)$ respectively) we get

$$
\left|\int_{a / \xi}^{1} x^{-\alpha-1} \cos \frac{1}{x} \frac{L_{a}(\xi x)}{L_{a}(\xi)} \cdot \frac{\xi x L_{a}^{\prime}(\xi x)}{L_{a}(\xi x)}\right| \leq \frac{a L^{\prime}(a)}{L(a)}\left|\int_{c_{1}}^{c_{2}} x^{-\alpha-1} \cos \frac{1}{x} d x\right|
$$

where $a / \xi<c_{1}<c_{2}<1$.
Since the integral $\int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} d x$ is convergent and

$$
\lim _{x \rightarrow+\infty} x \frac{L^{\prime}(x)}{L(x)}=0
$$

(11) holds if $a$ is fixed and large enough and $\xi>a$ is large enough. From $1^{\circ}$, $2^{\circ}$, and (5) it follows that

$$
\begin{equation*}
s_{n}\left(\int_{-1}^{1} H_{a}(x, y) \cdot d y\right) \sim 2 \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(n)}{(n \pi / 2)^{\alpha}} \tag{12}
\end{equation*}
$$

From $3^{\circ}$ we get

$$
\left\{\begin{array}{l}
s_{n}\left(\int_{-1}^{1} k_{a}(|x+y+2|) \cdot d y\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right),  \tag{13}\\
s_{n}\left(\int_{-1}^{1} k_{a}(|x+y-2|) \cdot d y\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right)
\end{array}\right.
$$

The function

$$
R(x, y)=k_{a}(|x-y-4|)+\sum_{n \neq 0 ; n \neq-1}\left(k_{a}(|x-y+4 n|)-k_{a}(|x+y+4 n+2|)\right)
$$

is bounded on $[-1,1]^{2}$ (a consequence of $4^{\circ}$ ), hence $\int_{-1}^{1} R(x, y) \cdot d y$ is a Hilbert Schmidt operator and

$$
\begin{equation*}
s_{n}\left(\int_{-1}^{1} R(x, y) \cdot d y\right)=\sigma\left(n^{-1 / 2}\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right) . \tag{14}
\end{equation*}
$$

From (12), (13), (14), and the Ky Fan Theorem it follows that

$$
\begin{equation*}
s_{n}\left(B_{a}^{\prime}\right) \sim 2 \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(n)}{(n \pi / 2)^{\alpha}} \tag{15}
\end{equation*}
$$

From (15) and from

$$
\int_{\mathbf{R}} k_{a}\left(\frac{|t|}{2}\right) e^{i t \xi} d t=2 \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(2 \xi)}{(2 \xi)^{\alpha}}(1+\sigma(1))
$$

by substitution in the eigenvalue relation $B_{a} e_{n}=\lambda_{n} e_{n}$, we obtain

$$
\begin{equation*}
s_{n}\left(B_{a}\right) \sim \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(n)}{(n \pi)^{\alpha}} \tag{16}
\end{equation*}
$$

Lemma 1 is proved.
From now on suppose $a$ is a fixed and large enough number such that (1) holds.
Proof of Theorem 1 in Case A. Since

$$
A f(x)=\int_{0}^{x} k(x-y) f(y) d y
$$

we have

$$
\left(A+A^{*}\right) f=B f=\int_{0}^{1} k(|x-y|) f(y) d y
$$

By Lemma 1 we get

$$
\lim _{n \rightarrow \infty} \frac{s_{n}(B)}{s_{n}\left(B_{a}\right)}=1
$$

and so

$$
s_{2 n}(B) \geq c_{1}^{\prime} s_{2 n}\left(B_{a}\right)
$$

( $c_{1}^{\prime}$ does not depend on $n$ ). The last inequality and (16) imply

$$
s_{2 n}(B) \geq c_{1} \frac{L(n)}{n^{\alpha}} \quad\left(c_{1} \text { does not depend on } n\right)
$$

Since $s_{2 n}(B) \leq s_{n}(A)+s_{n}\left(A^{*}\right)=2 s_{n}(A)$, we obtain

$$
\begin{equation*}
s_{n}(A) \geq \frac{c_{1}}{2} \frac{L(n)}{n^{\alpha}} \tag{17}
\end{equation*}
$$

Now we prove the following inequality

$$
\begin{equation*}
s_{n}(A) \leq \text { const } \frac{L(n)}{n^{\alpha}} \quad(\text { const does not depend on } n) . \tag{18}
\end{equation*}
$$

Here we use the following lemma proved in [4].

Lemma 2. Let $K_{n}(x, y)$ be a sequence of functions integrable to $x$ and to $y$ individually, $0 \leq x, y \leq 1$. Let $K(x, y)$ be a similar function, and suppose that for almost all $y$

$$
\int_{0}^{1}\left|K(x, y)-K_{n}(x, y)\right| d x \leq \beta_{n} \quad\left(\beta_{n} \rightarrow 0\right)
$$

and also that for almost all $x$

$$
\int_{0}^{1}\left|K(x, y)-K_{n}(x, y)\right| d y \leq \gamma_{n} \quad\left(\gamma_{n} \rightarrow 0\right)
$$

Finally, suppose that for each $n$

$$
\mathscr{K}_{n}=\int_{0}^{1} K_{n}(x, y) \cdot d y
$$

is a compact operator on $L^{2}(0,1)$. Then $\mathscr{K}=\int_{0}^{1} K(x, y) \cdot d y$ is also a compact operator on $L^{2}(0,1)$ and

$$
s_{n}(\mathscr{K}) \leq s_{n}\left(\mathscr{K}_{n}\right)+\sqrt{\beta_{n} \gamma_{n}} .
$$

Now let us put

$$
K_{n}(x, y)= \begin{cases}\left(x-y+\frac{1}{n}\right)^{\alpha-1} L\left(\frac{1}{x-y+1 / n}\right), & y<x \\ 0, & y \geq x\end{cases}
$$

and

$$
K(x, y)= \begin{cases}(x-y)^{\alpha-1} L\left(\frac{1}{x-y}\right), & y<x \\ 0, & y \geq x\end{cases}
$$

The function $t \mapsto t^{\alpha-1} L(1 / t)$ is decreasing (for $0<\alpha<1$ ) and hence

$$
\begin{aligned}
\int_{0}^{1}\left|K(x-y)-K_{n}(x, y)\right| d y & =\int_{0}^{1 / n} t^{\alpha-1} L\left(\frac{1}{t}\right) d t-\int_{x}^{x+1 / n} t^{\alpha-1} L\left(\frac{1}{t}\right) d t \\
& <\int_{0}^{1 / n} t^{\alpha-1} L\left(\frac{1}{t}\right) d t
\end{aligned}
$$

Since

$$
\int_{0}^{1 / n} L^{\alpha-1} L\left(\frac{1}{t}\right) d t=\int_{n}^{+\infty} t^{-\alpha-1} L(t) d t
$$

and

$$
\int_{x}^{+\infty} t^{-\alpha-1} L(t) d t \sim \frac{1}{\alpha} \frac{L(x)}{x^{\alpha}} \quad(x \rightarrow+\infty)
$$

we get

$$
\begin{equation*}
\int_{0}^{1}\left|K(x, y)-K_{n}(x, y)\right| d y \leq c_{3} \frac{L(n)}{n^{\alpha}} \quad\left(c_{3} \text { does not depend on } n\right) \tag{19}
\end{equation*}
$$

Similarly,
(20) $\quad \int_{0}^{1}\left|K_{n}(x, y)-K_{n}(x, y)\right| d x \leq c_{4} \frac{L(n)}{n^{\alpha}} \quad\left(c_{4}\right.$ does not depend on $\left.n\right)$.

From (19), (20), and Lemma 2 we obtain

$$
\begin{equation*}
s_{n}(A) \leq \sqrt{c_{3} c_{4}} \frac{L(n)}{n^{\alpha}}+s_{n}\left(\mathscr{K}_{n}\right) \tag{21}
\end{equation*}
$$

Now, we can estimate the norm $\left\|\mathscr{K}_{n}\right\|_{2}^{2}$ (Hilbert Schmidt norm). We have

$$
\begin{aligned}
\left\|\mathscr{K}_{n}\right\|_{2}^{2} & =\int_{0}^{1} \int_{0}^{1}\left|K_{n}(x, y)\right|^{2} d x d y \\
& =\int_{1 / n}^{1+1 / n} y^{2 \alpha-2}\left(L\left(\frac{1}{y}\right)\right)^{2} \cdot\left(1-y+\frac{1}{n}\right) d y \\
& \leq \int_{1 / n}^{1+1 / n} y^{2 \alpha-2}\left(L\left(\frac{1}{y}\right)\right)^{2} d y
\end{aligned}
$$

From this inequality by simple computation we get

$$
\left\|\mathscr{K}_{n}\right\|_{2}^{2} \leq c_{5} n^{1-2 \alpha}(L(n))^{2} \quad\left(c_{5} \text { does not depend on } n\right)
$$

Since $n s_{n}^{2}\left(\mathscr{K}_{n}\right) \leq\left\|\mathscr{K}_{n}\right\|_{2}^{2}$, we obtain

$$
\begin{equation*}
s_{n}\left(\mathscr{K}_{n}\right) \leq c_{6} \frac{L(n)}{n^{\alpha}} \quad\left(c_{6} \text { does not depend on } n\right) \tag{22}
\end{equation*}
$$

Now (18) follows from (21) and (22). The theorem is proved for the case when the function $L$ is not bounded.

Case B: The function $L$ is bounded. Since $L$ is nondecreasing we have $\lim _{x \rightarrow+\infty} L(x)=d<\infty$. By assumption of Theorem 1 we get $d>0$.

Lemma 3. Suppose $r \in C[0,1], r(0)=0$, and $G$ is a linear operator on $L^{2}(0,1)$ defined by

$$
G f(x)=\int_{0}^{x}(x-y)^{\alpha-1} r(x-y) f(y) d y
$$

If $0<\alpha<1 / 2$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{\alpha} s_{n}(G)=0 \tag{23}
\end{equation*}
$$

Proof of Lemma 3. Let us represent $G$ as

$$
G f(x)=\int_{0}^{1}|x-y|^{\alpha-1} M(x, y) f(y) d y
$$

where

$$
M(x, y)= \begin{cases}r(x-y), & 0<y \leq x<1 \\ 0, & 1 \geq y \geq x \geq 0\end{cases}
$$

Let $\varepsilon>0$. Then there exists $\delta>0$ such that $|M(x, y)|<\varepsilon$ if $|x-y|<\delta$. Put

$$
\Omega_{1}=[0,1]^{2} \backslash\{(x, y):|x-y|<\delta\}, \quad \Omega_{2}=[0,1]^{2} \backslash \Omega_{1}
$$

Suppose $G_{1}, G_{2}$ are linear operators on $L^{2}(0,1)$ defined by

$$
G_{i} f(x)=\int_{0}^{1}|x-y|^{\alpha-1} \chi_{\Omega_{i}}(x, y) M(x, y) f(y) d y, \quad i=1,2
$$

( $\chi_{\Omega_{i}}$ are characteristic functions of $\Omega_{i}, i=1,2$ ).

Then $G=G_{1}+G_{2}$ and

$$
\begin{equation*}
s_{2 n}(G) \leq s_{n}\left(G_{1}\right)+s_{n}\left(G_{2}\right) \tag{24}
\end{equation*}
$$

By Lemma 1 from [1] we obtain

$$
s_{n}\left(G_{1}\right) \leq \operatorname{const} \cdot \varepsilon\left[\int_{0}^{1 / n} t^{\alpha-1} d t+n^{-1 / 2}\left(\int_{1 / n}^{\infty} t^{2 \alpha-2} d t\right)^{1 / 2}\right]
$$

i.e. (since $0<\alpha<1 / 2$ ),

$$
\begin{equation*}
s_{n}\left(G_{1}\right) \leq \operatorname{const} \cdot \varepsilon \cdot \frac{1}{n^{\alpha}} \quad(\text { const does not depend on } n) \tag{25}
\end{equation*}
$$

On the other hand, $G_{2}$ is a Hilbert Schmidt operator and hence

$$
s_{n}\left(G_{2}\right) \leq c_{7}(\delta) \cdot n^{-1 / 2}
$$

From the previous inequality we get (for $0<\alpha<1 / 2$ )

$$
\begin{equation*}
n^{\alpha} s_{n}\left(G_{2}\right)<\varepsilon \tag{26}
\end{equation*}
$$

if $n$ is large enough.
From (24), (25), and (26) we obtain

$$
\lim _{n \rightarrow \infty} n^{\alpha} s_{2 n}(G)=0
$$

and

$$
\lim _{n \rightarrow \infty} n^{\alpha} s_{n}(G)=0
$$

Proof of Theorem 1 in Case B. Put $r(x)=L(1 / x)-d$. Applying Lemma 3 we get

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{\alpha} s_{n}\left(\int_{0}^{x}(x-y)^{\alpha-1}\left(L\left(\frac{1}{x-y}\right)-d\right) \cdot d y\right)=0 \tag{27}
\end{equation*}
$$

In [2] it is proved that

$$
s_{n}\left(\int_{0}^{x}(x-y)^{\alpha-1} \cdot d y\right) \sim \Gamma(\alpha)(n \pi)^{-\alpha}
$$

From (27), the previous asymptotic formula, and the Ky Fan Theorem we conclude

$$
s_{n}\left(\int_{0}^{x}(x-y)^{\alpha-1} L\left(\frac{1}{x-y}\right) \cdot d y\right) \sim d \cdot \Gamma(\alpha)(n \pi)^{-\alpha} .
$$

Theorem 1 is proved.
Remark. From the proof it is evident that if $L$ is bounded, then it is enough to suppose that $L$ is continuous and $\lim _{x \rightarrow \infty} L(x)=d \neq 0$.

Theorem 2. Suppose function $L$ satisfies conditions from the beginning of this paper. Let $r \in C^{1}[0,1], r(0)=0, k_{1}(x)=k(x) \cdot(1+r(x)) \quad\left(k(x)=x^{\alpha-1} L(1 / x)\right)$, and let $A_{1}: L^{2}(0,1) \rightarrow L^{2}(0,1)$ be a linear operator defined by

$$
A_{1} f(x)=\int_{0}^{x} k_{1}(x-y) f(y) d y
$$

If $0<\alpha<1 / 2$, then $s_{n}\left(A_{1}\right) \asymp L(n) / n^{\alpha}$.

Lemma 4. Suppose $A$ and $B$ are composed operators on Hilbert space $\mathscr{H}$ such that $s_{n}(A) \asymp L(n) / n^{\beta} \quad(L$ is a slowly varying function, $\beta>0)$ and $\lim _{n \rightarrow \infty} \frac{n^{\beta}}{L(n)} s_{n}(B)=0$. Then $s_{n}(A+B) \asymp L(n) / n^{\beta}$.
Proof of Lemma 4. From conditions $s_{n}(A) \asymp L(n) / n^{\beta}$ it follows that there exists constants $d_{1}>0$ and $d_{2}>0$ such that

$$
\begin{equation*}
d_{2} \frac{L(n)}{n^{\beta}} \leq s_{n}(A) \leq d_{1} \frac{L(n)}{n^{\beta}} . \tag{28}
\end{equation*}
$$

For arbitrary $k \in \mathbb{N}, n=(k+1) m+j, j=0,1,2, \ldots, k$, by properties of singular values [5], we have

$$
s_{(k+1) m+j}(A+B) \leq s_{k m+j}(A)+s_{m+1}(B),
$$

i.e.,

$$
\frac{s_{(k+1) m+j}(A+B)}{s_{(k+1) m+j}(A)} \leq\left(1+\frac{s_{m+1}(B)}{s_{k m+j}(A)}\right) \cdot \frac{s_{k m+j}(A)}{s_{(k+1) m+j}(A)} .
$$

From (28) we get

$$
\frac{s_{(k+1) m+j}(A+B)}{s_{(k+1) m+j(A)}} \leq\left(1+\frac{s_{m+1}(B)}{s_{k m+j}(A)}\right) \cdot \frac{d_{1}}{d_{2}}\left(\frac{(k+1) m+j}{k m+j}\right)^{\beta} \frac{L(k m+j)}{L((k+1) m+j)} .
$$

Since $\frac{n^{\beta}}{L(n)} s_{n}(A) \rightarrow 0$ (or equivalently $s_{n}(B) / s_{n}(A) \rightarrow 0$ ) we obtain

$$
\varlimsup_{n \rightarrow \infty} \frac{s_{n}(A+B)}{s_{n}(A)} \leq \frac{d_{1}}{d_{2}}\left(\frac{k+1}{k}\right)^{\beta} .
$$

As $k$ is arbitrary, we get

$$
\varlimsup_{n \rightarrow \infty} \frac{s_{n}(A+B)}{s_{n}(A)} \leq \frac{d_{1}}{d_{2}} .
$$

Similarly, we get

$$
\underline{\lim }_{n \rightarrow \infty} \frac{s_{n}(A+B)}{s_{n}(A)} \geq \frac{d_{2}}{d_{1}} .
$$

Lemma 4 is proved.
Proof of Theorem 2. Since $r \in C^{1}[0,1]$ and $r(0)=0, \int_{0}^{x} k(x-y) r(x-y) \cdot d y$ is a Hilbert Schmidt operator and therefore
(29) $s_{n}\left(\int_{0}^{x} k(x-y) r(x-y) \cdot d y\right)=\sigma\left(n^{-1 / 2}\right)=\sigma\left(\frac{L(n)}{n^{\alpha}}\right) \quad\left(0<\alpha<\frac{1}{2}\right)$.

From Theorem 1 we have

$$
\begin{equation*}
s_{n}\left(\int_{0}^{x} k(x-y) \cdot d y\right) \asymp \frac{L(n)}{n^{\alpha}} . \tag{30}
\end{equation*}
$$

The statement of Theorem 2 follows from (29), (30), and Lemma 4.
Example. Let $L(x)=(\ln x)^{\beta}, \beta \geq 0$, and let the function $r$ satisfy $r \in$ $C^{1}[0,1], r(0) \neq 0$. We consider the operator $T: L^{2}(0,1) \rightarrow L^{2}(0,1)$ defined by

$$
T f(x)=\int_{0}^{x}(x-y)^{\alpha-1}(-\ln (x-y))^{\beta} r(x-y) f(y) d y \quad(0<\alpha<1 / 2) .
$$

Then $s_{n}(T) \asymp(\ln n)^{\beta} / n^{\alpha}$.

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