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A LOWER BOUND FOR THE CLASS NUMBERS OF ABELIAN ALGEBRAIC NUMBER FIELDS WITH ODD DEGREE

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ABSTRACT. Let Δ_K, h_K, R_K denote the discriminant, the class number, and the regulator of the Abelian algebraic number field $K = \mathbb{Q}(\alpha)$ with degree d, respectively. In this note we prove that if $d > 1$, $2 \nmid d$, and the defining polynomial of α has exactly r_1 real zeros and r_2 pairs of complex zeros, then $h_K > w\sqrt{|\Delta_K|}/2^{r_1}(2\pi)^{r_2} 33 R_K \log 4|\Delta_K|$, where w is the number of roots of unity in K.

Let Δ_K, h_K, R_K denote the discriminant, the class number, and the regulator of the Abelian algebraic number field $K = \mathbb{Q}(\alpha)$ with degree d, respectively. In this note we prove the following result:

Theorem. *If $d > 1$, $2 \nmid d$, and the defining polynomial of α has exactly r_1 real zeros and r_2 pairs of complex zeros, then*

$$h_K > \frac{w\sqrt{|\Delta_K|}}{2^{r_1}(2\pi)^{r_2} 33 R_K \log 4|\Delta_K|}, \tag{1}$$

where w is the number of roots of unity in K.

Upon applying the above theorem, we can improve some known results concerning the lower bound of h_K. For instance, Barrucand, Loxton, and Williams [1] proved that if $K = \mathbb{Q}(D^{1/3})$, where $D = n^3 + m$ is not a cube, m and n are nonzero integers with $3n^2 \equiv 0 \pmod{m}$, then

$$h_K > \frac{0.14|\Delta_K|^{1/4}}{\log(|\Delta_K|/3)\log(|\Delta_K|/27)}.$$

Notice that $r_1 = 1$, $r_2 = 1$, $w = 2$, and $R_K < 3\log(|\Delta_K|/3)$ in this case. By (1), we get a better lower bound as follows:

$$h_K > \frac{\sqrt{|\Delta_K|}}{198\pi\log(|\Delta_K|/3)\log 4|\Delta_K|}.$$

The proof of Theorem. Let $\zeta_K(s)$ denote Dedekind's ζ-function of K. By [3, §42], if $\sigma > 1$, then

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

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where $b_1 = 1$ and $b_n \geq 0$ for $n > 1$. Since $\zeta_K(2) \geq b_1$ and

$$(-1)^m \zeta_K^{(m)}(2) = \sum_{n=1}^{\infty} \frac{b_n (\log n)^m}{n^2} \geq 0$$

for $m > 0$, we have

$$\zeta_K(s) = \sum_{m=0}^{\infty} a_m (2-s)^m, \qquad a_0 \geq 1, a_m \geq 0, m > 0, |s-2| < 1. \tag{2}$$

Let X be the group of Dirichlet characters associated to K. It is a well-known fact that $\zeta_K(s) = \zeta(s)\xi_K(s)$, where $\zeta(s)$ is the Riemann ζ-function,

$$\xi_K(s) = \prod_{\substack{\chi \in X \\ \chi \neq \chi_0}} L(s, \chi),$$

where χ_0 is the trivial character and $L(s, \chi)$ is the L-series attached to the character χ. Since $\zeta_K(s)$ has only simple pole at $s = 1$ of residue $\xi_K(1)$, the function $g(s) = \zeta_K(s) - \xi_K(1)/(s-1)$ is regular. From (2), we get

$$g(s) = \sum_{m=0}^{\infty} (a_m - \xi_K(1))(2-s)^m. \tag{3}$$

For any $\sigma > 0$ and any $x \geq 1$, using Abel's transformation,

$$L(s, \chi) = \sum_{1 \leq n \leq x} \frac{\chi(n)}{n^s} - \frac{S(x, \chi)}{x^s} + s \int_x^{\infty} \frac{S(z, \chi)}{z^{s+1}}\, dz, \tag{4}$$

where $S(x, \chi) = \sum_{1 \leq n \leq x} \chi(n)$. Let f_χ denote the conductor of χ. By Pólya's theorem, $|S(x, \chi)| < \sqrt{f_\chi} \log f_\chi$. By (4), we get

$$|L(s, \chi)| \leq \sum_{1 \leq n \leq x} \frac{1}{n^\sigma} + \frac{2\sqrt{f_\chi} \log f_\chi}{x^\sigma} < 1 + \frac{x^{1-\sigma} - 1}{1-\sigma} + \frac{2\sqrt{f_\chi} \log f_\chi}{x^\sigma}. \tag{5}$$

Putting $x = \sqrt{f_\chi} \log f_\chi$. We get from (5) that

$$|L(s, \chi)| < 4 f_\chi^{1/4} \sqrt{\log f_\chi} < f_\chi^{5/4}, \qquad \sigma \geq \frac{1}{2}, \tag{6}$$

since $f_\chi \geq 5$. Furthermore, by the conductor-discriminant formula

$$\Delta_K = (-1)^{r_2} \prod_{\chi \in X} f_\chi, \tag{7}$$

we get from (6) that

$$|\xi_K(s)| < \left| \prod_{\substack{\chi \in X \\ \chi \neq \chi_0}} f_\chi^{5/4} \right| = |\Delta_K|^{5/4}, \qquad \sigma \geq \frac{1}{2}. \tag{8}$$

Simultaneously, since $|\zeta(s)| \leq 1/|s-1| + |s|/\sigma$ for $\sigma > 0$, we have

$$|\zeta(s)| \leq 3, \qquad |s-2| = \frac{3}{2}.$$

Therefore, by (8),

$$|\zeta_K(s)| < 3|\Delta_K|^{5/4}, \qquad |s-2| = \frac{3}{2}. \tag{9}$$

Further, by a well-known fact that $|L(1, \chi)| < \log f_\chi + 2$, we get from (9) that

$$|g(s)| \le |\zeta_K(s)| + \left|\frac{\xi_K(1)}{s-1}\right| < 4|\Delta_K|^{5/4} \tag{10}$$

for $|s-2| = 3/2$. Furthermore, by the maximum modulus principle, (10) holds for $|s-2| \le 3/2$. Using Cauchy's theorem, we find from (3) and (10) that

$$|a_m - \xi_K(1)| < 4|\Delta_K|^{5/4}\left(\frac{2}{3}\right)^m, \qquad m \ge 0. \tag{11}$$

Let M be an integer with $M > 1$. By (2) and (11), if $13/14 < \sigma < 1$, then

$$\begin{aligned} g(\sigma) &= \zeta_K(\sigma) - \frac{\xi_K(1)}{\sigma - 1} \\ &\le \sum_{m=0}^{M-1}(a_m - \xi_K(1))(2-\sigma)^m - \sum_{m=M}^{\infty}|a_m - \xi_K(1)|(2-\sigma)^m \\ &< 1 - \xi_K(1)\sum_{m=0}^{M-1}(2-\sigma)^m - 4|\Delta_K|^{5/4}\sum_{m=M}^{\infty}\left(\frac{2}{3}(2-\sigma)\right)^m \\ &< 1 - \xi_K(1)\frac{(2-\sigma)^M - 1}{1-\sigma} - 14|\Delta_K|^{5/4}\left(\frac{5}{7}\right)^M. \end{aligned} \tag{12}$$

Put

$$M = \left[\frac{\log(140|\Delta_K|^{5/4})}{\log(7/5)}\right] + 1. \tag{13}$$

We get from (12) and (13) that

$$\zeta_K(\sigma) > \frac{9}{10} - \frac{(2-\sigma)^M}{1-\sigma}\xi_K(1). \tag{14}$$

By a recent result of Chen and Wang [2], if χ is a complex character, then $L(s, \chi)$ has no zero in the range

$$1 \ge \sigma > 1 - \frac{c}{\log f_\chi(|t|+2)}, \qquad t \ge 0,$$

where

$$c = \max\left(0.089193, \frac{19.09712}{43.14093 + 12.169/\log f_\chi(|t|+2)} - 0.339\right).$$

Since $c > 0.0553581$ for $f_\chi \ge 5$, we see that $\angle(\sigma, \chi) \ne 0$ for the range $1 - 1/18.0642\log 2f_\chi \le \sigma < 1$. Since $2 \nmid d$, all characters of X are complex characters. Notice that $d \ge 3$. We get from (7) that $|\Delta_K| \ge f_\chi^2$ for any $\chi \in X$. Hence, $\angle(\sigma, \chi) \ne 0$ for $1 - 1/9.0321\log 4|\Delta_K| \le \sigma < 1$. It implies that $\zeta_K(\sigma) < 0$ for $1 - 1/9.0321\log 4|\Delta_K| \le \sigma < 1$, and by (14), we obtain

$$\xi_K(1) > \frac{9(1-\sigma_0)}{10(2-\sigma_0)^M}, \tag{15}$$

where $\sigma_0 = 1 - 1/9.0321 \log 4|\Delta_K|$. Since $d \geq 3$ and $|\Delta_K| \geq 23$, we get from (13) that

$$\begin{aligned}\log(2-\sigma_0)^M &= M \log\left(1 + \frac{1}{9.0321 \log 4|\Delta_K|}\right) \\ &< \left(1 + \frac{\log 140|\Delta_K|^{5/4}}{\log(7/5)}\right)\left(\frac{1}{9.0321 \log 4|\Delta_K|}\right) \\ &\leq \left(1 + \frac{\log 140 + \log 23^{5/4}}{\log(7/5)}\right)\left(\frac{1}{9.0321 \log 4.23}\right) < 1.17\end{aligned}$$

and $(2-\sigma_0)^M < 3.23$. Substituting it into (15),

$$\xi_K(1) > \frac{1}{33 \log 4|\Delta_K|}. \tag{16}$$

Thus, by (16) and the class number formula

$$h_K = \frac{w\sqrt{|\Delta_K|}}{2^{r_1}(2\pi)^{r_2} R_K} \xi_K(1),$$

we get (1) immediately. The theorem is proved.

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