THE GAUSSIAN-WAHL MAP FOR TRIGONAL CURVES

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(Communicated by Eric Friedlander)

ABSTRACT. If a curve C is embedded in projective space by a very ample line bundle L , the Gaussian map $\Phi_{C,L}$ is defined as the pull-back of hyperplane sections of the classical Gauss map composed with the Plücker embedding. When $L = K$, the canonical divisor of the curve C , the map is known as the Gaussian-Wahl map for C . We determine the corank of the Gaussian-Wahl map to be $g + 5$ for all trigonal curves (i.e., curves which admit a 3-to-1 mapping onto the projective line) by examining the way in which a trigonal curve is naturally embedded in a rational normal scroll.

1. INTRODUCTION

(1.1) If L is a line bundle on a projective variety X , we can define the Gaussian map

$$\Phi_L : \Lambda^2 H^0(X, L) \rightarrow H^0(X, \Omega_X^1 \otimes L^2)$$

essentially by $\Phi_L(f \wedge g) = f dg - g df$. The name Gaussian derives from the case where X is a curve and L is very ample; in this case Φ_L is simply the pull-back of hyperplane sections of the classical Gauss map, after composing with the Plücker embedding. When X is a curve and $L = K_X$, the canonical divisor of the curve, Wahl proved the following remarkable theorem concerning the map Φ_K , which we refer to as the Gaussian-Wahl map.

Theorem 1.2 (Wahl [Wa]). *If a non-singular curve lies on a K3 surface, then its Gaussian-Wahl map Φ_K is not surjective.*

Other authors have shown that the Gaussian-Wahl map is surjective for the general curve of sufficiently high genus [CHM], and have studied the corank of the map for certain families of curves for which the map is not surjective. Ciliberto and Miranda have shown that $\text{corank}(\Phi_K) = g + 5$ for the general trigonal curve of genus $g \geq 4$ [CM]. They went on to exhibit a specific class of cyclic trigonal curves, defined by the equation $y^3 = x^{3L} - 1$, for which the corank of Φ_K is equal to this same generic value. We prove in this paper that in fact this is the case for *every* trigonal curve of genus $g \geq 4$. After some preliminary discussion we prove the main theorem in §3.

Received by the editors September 9, 1993; the contents of this paper were presented at the Joint Mathematics Meetings of the AMS on January 14, 1994.

1991 *Mathematics Subject Classification*. Primary 14H45; Secondary 14J26.

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 0002-9939/95 \$1.00 + \$.25 per page

Theorem 1.3. *The corank of the Gaussian-Wahl map is $g + 5$ for every nonsingular trigonal curve of genus $g \geq 4$.*

2. TRIGONAL CURVES

(2.1) In this section we recall several well-known facts about trigonal curves and rational normal scrolls which we will use in the proof of the main theorem (cf. [Co, Ma]). A non-singular trigonal curve C of genus g , canonically embedded in $\mathbf{P}^{g-1}$, lies on a rational normal scroll whose invariants are determined by the g_3^1 . Specifically, if k is the unique positive integer such that $h^0((k+1)g_3^1) = k+2$ and $h^0((k+2)g_3^1) \geq k+4$, then C lies on the scroll $S_{k,l} \subset \mathbf{P}^{k+l+1}$, where $k+l = g-2$ and $k \leq l$. Maroni showed that this invariant k is bounded by

$$\frac{g-1}{d} \leq k+1 \leq \frac{g}{2}$$

and that the ruling of the scroll cuts out the g_3^1 on the curve [Ma]. Where convenient we will denote the scroll simply as S .

(2.2) The scroll $S_{k,l}$, which is isomorphic to the surface F_n , $n = l - k$, has Picard group generated by a hyperplane section H and a fibre of the ruling R , with the following intersections: $H^2 = k+l$, $H \cdot R = 1$, and $R^2 = 0$. If $k < l$, there is a unique curve B of negative self-intersection on the scroll with class $B \equiv H - lR$ and intersections $B^2 = -(l-k)$ and $B \cdot R = 1$. The class of the curve C and the canonical divisor of the scroll K_S are easily computed, as are those of two other divisors that we will use later:

$$\begin{aligned} C &\equiv 3H + (4-g)R, \\ K_S &\equiv -2H + (g-4)R \equiv -2B - (n+2)R, \\ K_S + C &\equiv H \equiv B + lR, \\ 2K_S + C &\equiv -H + (g-4)R \equiv -B + (k-2)R. \end{aligned}$$

(2.3) We collect the following computations about global sections of sheaves over the scroll S (cf. [DM 3.2, 3.3]). If $D \equiv rB + sR$ is a divisor on a rational normal scroll $S_{k,l} \cong F_{l-k} = F_n$, then

- (i) $H^0(S, \mathcal{O}_S(D)) = 0 = H^0(S, \Omega_S^1(D))$ if $r < 0$ or $s < 0$.
- (ii) $h^0(S, \Omega_S^1(D)) = 2rs - nr^2 - 2$ if $r \geq 1$ and $s \geq nr + 1$.

3. THE PROOF OF THE THEOREM

(3.1) If the trigonal curve C has genus $g = 4$, then it is a complete intersection curve, which we handle with a separate argument at the end of the proof; for the remainder, we assume $g \geq 5$. From the inclusion $C \subset S$, we have the following diagram involving the Gaussian surface map $\Phi_{S, K+C}$:

$$\begin{array}{ccc} \Lambda^2 H^0(S, K_S + C) & \xrightarrow{\Phi_{S, K+C}} & H^0(S, \Omega_S^1(2K_S + 2C)) \\ \text{Res} \downarrow & & \downarrow \gamma \\ \Lambda^2 H^0(C, \Omega_C^1) & \xrightarrow{\Phi_K} & H^0(C, \Omega_C^{1 \otimes 3}) \end{array}$$

First notice that the restriction map $H^0(S, K_S + C) \rightarrow H^0(C, \Omega_C^1)$ is surjective because the scroll S is a regular surface; consequently the left vertical map Res is surjective as well. We proceed to prove the theorem by showing that the top map is surjective and the right vertical map $\gamma : H^0(S, \Omega_S^1(2K_S + 2C)) \rightarrow H^0(C, \Omega_C^{1\otimes 3})$ is injective. The rank of the Gaussian-Wahl map Φ_K is then equal to the dimension of the space $H^0(S, \Omega_S^1(2K_S + 2C))$, which we easily compute to be $4g - 10$, and the theorem follows.

Proposition 3.2. $\Phi_{S, K+C}$ is surjective with $\text{rank} = 4g - 10$.

Proof. Duflot and Miranda have calculated when the Gaussian map $\Phi_{S, D}$ is surjective for a divisor D on a rational normal scroll $S_{k, l} \cong \mathbf{F}_{l-k} = \mathbf{F}_n$ by explicitly writing bases for $H^0(S, \mathcal{O}_S(D))$ and $H^0(S, \Omega_S^1(2D))$. The first part of the proposition follows immediately from the following result [DM, 4.5].

Theorem 3.3 (Duflot-Miranda). *The Gaussian surface map*

$$\Phi_{S, D} : H^0(S, \mathcal{O}_S(D)) \rightarrow H^0(S, \Omega_S^1(2D))$$

is surjective for a divisor $D \equiv rB + sR$ on $S_{k, l} \cong \mathbf{F}_n$ if $r \geq 0$ and $s \geq nr + 1$.

In our case, $D \equiv K_S + C \equiv B + lR$, so $r = 1 \geq 0$ and $s = l$. If $g \geq 5$, then Maroni's inequality shows that $k \geq 1$; consequently $nr + 1 = l - k + 1 \leq l = s$ and the theorem applies. Therefore the rank of $\Phi_{S, K+C}$ is simply the dimension of the target space, which we compute using (2.3):

$$\begin{aligned} \text{rank}(\Phi_{S, K+C}) &= h^0(S, \Omega_S^1(2K_S + 2C)) \\ &= h^0(S, \Omega_S^1(2B + 2lR)) \\ &= 2(2)(2l) - (l - k)4 - 2 \\ &= 4(l + k) - 2 \\ &= 4g - 10 \end{aligned}$$

since $r = 2 \geq 1$ and $s = 2l \geq 2(l - k) + 2$. $\square$

Proposition 3.4. $\gamma : H^0(S, \Omega_S^1(2K_S + 2C)) \rightarrow H^0(C, \Omega_C^{1\otimes 3})$ is injective.

Proof. Consider the following two exact sequences associated with the log complex of $C \subset S$:

$$\begin{aligned} 0 \rightarrow \Omega_S^1 \rightarrow \Omega_S^1(\log C) \rightarrow \mathcal{O}_C \rightarrow 0, \\ 0 \rightarrow \Omega_S^1(\log C)(-C) \rightarrow \Omega_S^1 \rightarrow \Omega_C^1 \rightarrow 0. \end{aligned}$$

We twist and take global sections to construct the following commutative diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ H^0(S, \Omega_S^1(2K_S + C)) & \longrightarrow & H^0(S, \Omega_S^1(\log C)(2K_S + C)) & \longrightarrow & H^0(C, \mathcal{O}_C(2K_S + C)) \\ & & \downarrow & & \downarrow \\ H^0(S, \Omega_S^1(2K_S + C)) & \longrightarrow & H^0(S, \Omega_S^1(2K_S + 2C)) & \xrightarrow{\alpha} & H^0(C, \Omega_S^1(2K_S + 2C)|_C) \\ & & \downarrow \gamma & & \downarrow \beta \\ & & H^0(C, \Omega_C^{1\otimes 3}) & = & H^0(C, \Omega_C^{1\otimes 3}) \end{array}$$

The right vertical sequence is obtained from a twist of the standard conormal sequence

$$0 \rightarrow \mathcal{O}_C(-C) - \Omega_S^1|_C \rightarrow \Omega_C^1 \rightarrow 0.$$

We show that $\ker(\gamma) = H^0(S, \Omega_S^1(\log C)(2K_S + C)) = 0$.

(3.5) Let $\pi : S \rightarrow \mathbf{P}^1$ be the map associated with the $\mathbf{P}^1$ -bundle S , B , and R generators of $\text{Pic}(S)$ as above, and let Z be the 0-dimensional subscheme of S of ramification points of the restriction map $\pi|_C : C \rightarrow \mathbf{P}^1$. Since $K_S \equiv -2B - (n+2)R$ and $\Omega_{\mathbf{P}^1}^1 \cong \mathcal{O}_{\mathbf{P}^1}(-2)$, we have the following exact sequence:

$$0 \rightarrow \pi^* \Omega_{\mathbf{P}^1}^1 \rightarrow \Omega_S^1 \rightarrow \mathcal{O}_S(-2B - nR) \rightarrow 0.$$

Furthermore, since $\Omega_S^1(\log C)(-C) \subset \Omega_S^1$, we can construct the following diagram involving the ideal sheaf of Z :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^* \Omega_{\mathbf{P}^1}^1(-C) & \longrightarrow & \Omega_S^1(\log C)(-C) & \longrightarrow & \mathcal{I}_Z \cdot \mathcal{O}_S(-2B - nR) \longrightarrow 0 \\ & & & & \cap & & \cap \\ 0 & \longrightarrow & \pi^* \Omega_{\mathbf{P}^1}^1 & \longrightarrow & \Omega_S^1 & \longrightarrow & \mathcal{O}_S(-2B - nR) \longrightarrow 0 \end{array}$$

Twisting the top sequence by $2K_S + 2C \equiv 2B + 2lR$ and taking global sections gives

$$\begin{aligned} 0 &\rightarrow H^0(S, \pi^* \Omega_{\mathbf{P}^1}^1(2K_S + C)) \rightarrow H^0(S, \Omega_S^1(\log C)(2K_S + C)) \\ &\rightarrow H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g-2)R)). \end{aligned}$$

The first space is zero since

$$H^0(S, \pi^* \Omega_{\mathbf{P}^1}^1(-B + (k-2)R)) \cong H^0(S, \mathcal{O}_S(-B + (k-4)R)),$$

which is zero by (2.3). Now, since

$$H^0(S, \mathcal{O}_S((g-2)R)) \cong H^0(S, \pi^* \mathcal{O}_{\mathbf{P}^1}(g-2))$$

the former space can be represented by homogeneous polynomials of degree $\leq g-2$ over $\mathbf{P}^1$, while $H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g-2)R))$ is isomorphic to those polynomials which vanish on the scheme Z . The Riemann-Hurwitz formula shows that there are at least $g+2$ ramification points of the restriction map $\pi|_C : C \rightarrow \mathbf{P}^1$ (with equality if all are total ramification points); but no polynomial of degree $\leq g-2$ can vanish on $\geq g+2$ points. Therefore

$$H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g-2)R)) = 0, \quad H^0(S, \Omega_S^1(\log C)(2K_S + C)) = 0,$$

and the proposition is proved.

4. TRIGONAL CURVES OF GENUS $g = 4$

(4.1) To finish the proof of the theorem, we show that $\text{corank}(\Phi_K) = g+5 = 9$ for the trigonal curves of genus 4, each of which is the complete intersection of a quadric and a cubic surface in $\mathbf{P}^3$. Accordingly we construct the following

diagram involving the Gaussian map Φ for the divisor $\mathcal{O}_{\mathbf{P}^3}(1)$ on $\mathbf{P}^3$:

$$\begin{array}{ccc}
 \Lambda^2 H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)) & \xrightarrow{\Phi} & H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}^1(2)) \\
 \text{Res} \downarrow & & \downarrow \gamma \\
 \Lambda^2 H^0(C, \Omega_C^1) & \xrightarrow{\Phi_K} & H^0(C, \Omega_C^{1 \otimes 3})
 \end{array}
 \begin{array}{c}
 \nearrow \alpha \\
 \searrow \beta
 \end{array}
 \begin{array}{c}
 H^0(C, \Omega_{\mathbf{P}^3}^1(2)|_C)
 \end{array}$$

By a theorem of Wahl the Gaussian map for any divisor $\mathcal{O}_{\mathbf{P}^n}(k)$ on $\mathbf{P}^n$ is surjective ($n \geq 1$), so the top map Φ is surjective [Wa]. The left vertical map Res is again surjective due to the normality of $\mathbf{P}^3$, and the right map γ factors through the space $H^0(C, \Omega_{\mathbf{P}^3}^1(2)|_C)$. We show that $\text{Im}(\alpha) \cap \ker(\beta) = 0$ and hence that the Gaussian-Wahl map has rank equal to the rank of the map α in the diagram above. We write $\{x_j dx_i - x_i dx_j \mid 0 \leq i < j \leq 3\}$ as a 6-dimensional basis for the space $H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}^1(2))$. An element of $\text{Im}(\alpha) \cap \ker(\beta)$ can be expressed as a quadric polynomial Q that vanishes along the curve C such that

$$dQ \equiv \sum_{0 \leq i < j \leq 3} \lambda_{ij}(x_j dx_i - x_i dx_j) \pmod{I_C},$$

where $\lambda_{ij} \in \mathbb{C}$. Since C is not contained in any hyperplane of $\mathbf{P}^3$, this means

$$\frac{\partial Q}{\partial x_i} = \sum_{j>i} \lambda_{ij} x_j - \sum_{j<i} \lambda_{ji} x_i \quad \forall i,$$

and $2Q = 0$ by Euler's lemma. Therefore, $\text{Im}(\alpha) \cap \ker(\beta) = 0$, $\text{rank}(\Phi_K) = \text{rank}(\alpha) = 6$, and $\text{corank}(\Phi_K) = 9$. $\square$

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