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ON $\mathscr{M}$-HARMONIC BLOCH SPACE

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ABSTRACT. We show that many of the characterizations of analytic Bloch functions also characterize $\mathscr{M}$-harmonic Bloch functions.

1. INTRODUCTION

The class of analytic Bloch functions on the unit disc and the unit ball B in C^n is well known, and it has been studied by many authors ([3], [4], [5], [6], [8], [13], [14]). In this note $\mathscr{M}$-harmonic Bloch functions on B are studied. Our results show that many of the characterizations of analytic Bloch functions also characterize $\mathscr{M}$-harmonic Bloch functions. Some other characterizations of $\mathscr{M}$-harmonic Bloch functions are given in [9].

To state our main result we need some notation. As in [12], we say that a function $u \in C^2(B)$ is $\mathscr{M}$-harmonic in B, $u \in \mathscr{M}$, if $\widetilde{\Delta} u(z) = 0$ for every $z \in B$. The operator $\widetilde{\Delta}$ is the invariant Laplacian defined by $\widetilde{\Delta} u(z) = \Delta(u \circ \varphi_z)(0)$, $z \in B$, where Δ is the ordinary Laplacian and φ_z is the standard automorphism of B taking 0 to z (see [12]).

For $f \in C^1(B)$, $Df = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ denotes the complex gradient of f, and $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_{2n})$, $z_k = x_{2k-1} + i x_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f.

For $f \in C^1(B)$ let $\widetilde{D} f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f, respectively.

If $f \in C^1(B)$ let

$$|\nabla_T f(z)|^2 = 2(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2), \qquad z \in B,$$

be the tangential gradient of f. As usual, R denotes the radial derivative $R = \sum_{j=1}^n z_j \partial/\partial z_j$.

We say that $f \in \mathscr{M}$ is $\mathscr{M}$-harmonic Bloch function, $f \in \mathscr{M}\mathscr{B}$, if $\|f\|_{\mathscr{B}} = \sup_{z \in B} |\widetilde{\nabla} f(z)| < \infty$.

We define the little $\mathscr{M}$-harmonic Bloch space $\mathscr{M}\mathscr{B}_0$ to be the subspace of $\mathscr{M}\mathscr{B}$ for which $\lim_{|z| \to 1} |\widetilde{\nabla} f(z)| = 0$.

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Theorem 1. *Let $f \in \mathscr{M}$. Then the following are equivalent:*

(1) f *is a $\mathscr{M}$-harmonic Bloch function,*
(2) $\sup_{z\in B}(\widetilde{\Delta}|f|^2)^{1/2} < \infty$,
(3) $\sup_{z\in B}(1-|z|^2)^{1/2}|\nabla_T f(z)| < \infty$,
(4) $\sup_{z\in B}(1-|z|^2)|\nabla f(z)| < \infty$,
(5) $\sup_{z\in B}(1-|z|^2)(|Rf(z)|+|\overline{R}f(z)|) < \infty$, *where* $\overline{R} = \sum_{j=1}^n \overline{z_j}\partial/\partial\overline{z_j}$.

In [14] Theorem 1 was proved for analytic functions. The proof, based on the Cauchy integral formula, shows that, if $f : B \mapsto C$ is analytic and $|\nabla f(z)|$ grows at most as fast as $1/(1-|z|^2)$, then the directional derivatives of f in directions perpendicular to the radial directions grow at most as fast as $1/(1-|z|^2)^{1/2}$. Using the integral representation formulas for derivatives of $\mathscr{M}$-harmonic functions obtained in [1] we show that $\mathscr{M}$-harmonic functions also behave twice as well in the complex-tangential directions.

The equivalences of Theorem 1 carry over to the little $\mathscr{M}$-harmonic Bloch space as is shown in the following theorem.

Theorem 2. *Let $f \in \mathscr{M}$. Then the following statements are equivalent:*

(1) $f \in \mathscr{M}\mathscr{B}_0$,
(2) $(\widetilde{\Delta}|f|^2(z))^{1/2} = o(1)$, $|z| \to 1$,
(3) $|\nabla_T f(z)| = o(1/\sqrt{1-|z|})$, $|z| \to 1$,
(4) $|\nabla f(z)| = o(1/(1-|z|))$, $|z| \to 1$,
(5) $(1-|z|^2)(|Rf(z)|+|R\bar{f}(z)|) = o(1)$, $|z| \to 1$.

We omit details.

For $f \in \mathscr{M}$ let

$$\partial f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z), \frac{\partial f}{\partial \overline{z_1}}(z), \dots, \frac{\partial f}{\partial \overline{z_n}}(z)\right)$$

and for any positive integer m we write $\partial^m f(z) = (\partial^\alpha\bar{\partial}^\beta f(z))_{|\alpha|+|\beta|=m}$ and $|\partial^m f(z)|^2 = \sum_{|\alpha|+|\beta|=m} |\partial^\alpha\bar{\partial}^\beta f(z)|^2$, where

$$\partial^\alpha\bar{\partial}^\beta f(z) = \frac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1}, \dots, \partial z_n^{\alpha_n} \partial \overline{z_1}^{\beta_1}, \dots, \partial \overline{z_n}^{\beta_n}},$$

α and β are multi-indices.

Our second result is the following theorem which relates the Bloch norm of an $\mathscr{M}$-harmonic function with quantities involving integrals of the higher-order derivative of the function. Even though $\|f\|_{\mathscr{B}}$, $f \in \mathscr{M}$, is not a norm, we refer to $\|f\|_{\mathscr{B}}$ as the Bloch norm of the function f. The quantity $|f(0)| + \|f\|_{\mathscr{B}}$ defines a norm on the linear space $\mathscr{M}$ which, equipped with this norm, is a Banach space.

Theorem 3. *Let $0 < p < \infty$, $0 < r < 1$, and $m \in N$. Then for a $\mathscr{M}$-harmonic function f the following quantities are equivalent:*

(i) $\|f\|_{\mathscr{B}} < \infty$,
(ii) $\sup_{z\in B}(1-|z|)|\partial f(z)| < \infty$,
(iii) $\sup_{z\in B}(1-|z|)^m|\partial^m f(z)| < \infty$,
(iv) $\sup_{z\in B}\int_{E_r(z)} |\partial^m f(w)|^p(1-|w|)^{mp-n-1}\,dv(w) < \infty$.

For analytic functions Theorem 3 was proved in [6], [13].

2. Proof of Theorem 1

For $a \in B$ and $0 < r < 1$ let $E_r(a) = \{z \in B : |\varphi_a(z)| < r\}$. The measure τ defined on B by $d\tau(z) = (1-|z|^2)^{-n-1}\,d\nu(z)$, where ν denotes the $2n$-dimensional Lebesgue measure on B normalized so that $\nu(B) = 1$, is $\mathscr{M}$-invariant (see [12]). In particular, $\tau(E_r(a)) = \tau(rB)$, $a \in B$, $0 < r < 1$. Any unexplained notation is as in [12].

Lemma 2.1. *Let $0 < r < 1$. There is a constant C such that if $f \in \mathscr{M}$, then*
(a) $|T_{ij}Rf(w)| \le C(1-|w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)|\,d\tau(z)$, $w \in B$,
(b) $|T_{ij}\overline{R}f(w)| \le C(1-|w|^2)^{-1/2} \int_{E_r(w)} |\overline{R}f(z)|\,d\tau(z)$, $w \in B$.
As usual, $T_{ij} = \overline{z_i}\partial/\partial z_j - \overline{z_j}\partial/\partial z_i$ are tangential derivatives.

Here and elsewhere constants are denoted by C which may indicate a different constant from one occurrence to the next.

Proof. (a) By the formula (1.3) in [1]

$$Rf(w) = \int_S \frac{Rf(\varphi_w(\rho\xi))}{1-\langle \rho\xi, w\rangle}\,d\sigma(\xi), \qquad w \in B,\ 0 < \rho < 1.$$

Multiplying this equality by $2n\rho^{2n-1}(1-\rho^2)^{-n-1}h(\rho)\,d\rho$, where h is a radial function which belongs to $C^\infty(B)$ with compact support in B such that $\int_B h(z)\,d\tau(z) = 1$, and then integrating from 0 to 1 and using the invariance of the measure τ, we get

$$\begin{aligned} Rf(w) &= \int_B h(\varphi_w(z)) \frac{1}{1-\langle \varphi_w(z), w\rangle} Rf(z)\,d\tau(z) \\ &= \int_B h(\varphi_z(w)) \frac{1-\langle z, w\rangle}{1-|w|^2} Rf(z)\,d\tau(z), \end{aligned}$$

by Theorem 2.2.5 ([12], p. 28).

Denote the components of φ_z by $\varphi_1(\cdot, z), \dots, \varphi_n(\cdot, z)$. Since these are holomorphic in B with $\sup_{z,w\in B} |\varphi_m(z, w)| = 1$, $1 \le m \le n$, we have $|T_{ij}\varphi_m(w, z)| \le C(1-|w|^2)^{-1/2}$, by Lemma 2.3 in [2] (see also [10]).

Note that $T_{ij}(1-\langle z, w\rangle)/(1-|w|^2) = 0$ (here the operator T_{ij} denotes differentiation with respect to w).

Now the chain rule gives

$$\begin{aligned} |T_{ij}Rf(w)| &= \left| \int_B h'(\varphi_z(w)) \left[\sum_{m=1}^n \frac{\overline{\varphi_m}(w, z)}{2|\varphi_z(w)|} T_{ij}\varphi_m(w, z) \right] \right. \\ &\qquad \left. \cdot \frac{1-\langle z, w\rangle}{1-|w|^2} Rf(z)\,d\tau(z) \right| \\ &\le C(1-|w|^2)^{-1/2} \int_B |h'(\varphi_z(w))| \frac{|1-\langle z, w\rangle|}{1-|w|^2} |Rf(z)|\,d\tau(z). \end{aligned}$$

By a suitable choice of a function h we obtain

$$|T_{ij}Rf(w)| \le C(1-|w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)|\,d\tau(z), \quad \text{for some } 0 < r < 1.$$

Here, we have used the fact that $|1-\langle z, w\rangle| \cong 1-|w|^2$, if $z \in E_r(w)$.

(b) Since $\bar{f} \in \mathscr{M}$ and $\overline{R}f = \overline{R\bar{f}}$, from the formula for Rf, obtained above, we get

$$\overline{R}f(w) = \int_B h(\varphi_z(w)) \frac{1 - \langle w, z\rangle}{1 - |w|^2} \overline{R}f(z)\, d\tau(z)$$

and consequently

$$\begin{aligned} |T_{ij}\overline{R}f(w)| \le \int_B |h'(\varphi_w(z))| &\left| \sum_{m=1}^n \frac{\overline{\varphi_m}(w, z)}{2|\varphi_z(w)|} T_{ij}\varphi_m(w, z) \right| \\ &\cdot \frac{|1 - \langle w, z\rangle|}{1 - |w|^2} |\overline{R}f(z)|\, d\tau(z) \\ + \int_B |h(\varphi_w(z))|\, |T_{ij}(1 - \langle w, z\rangle)| &\frac{|\overline{R}f(z)|}{1 - |w|^2}\, d\tau(z) = I_1 + I_2. \end{aligned}$$

Note that here we have used the fact that

$$T_{ij}\left(\frac{1 - \langle w, z\rangle}{1 - |w|^2}\right) = \frac{1}{1 - |w|^2} T_{ij}(1 - \langle w, z\rangle).$$

If the operator T_{ij} denotes differentiation with respect to w as above, and $z \in E_r(w)$ is written as $z = \varphi_w(u)$ (with $u \in rB$), then it is easily seen that

$$|T_{ij}(1 - \langle w, z\rangle)| = \left| \frac{S_w(u_i w_j - u_j w_i)}{1 - \langle u, w\rangle} \right| \le \frac{2r}{1 - r} S_w = \frac{2r}{1 - r}(1 - |w|^2)^{1/2}.$$

Hence

$$I_2 \le \frac{2r}{1 - r}(1 - |w|^2)^{-1/2} \int_{E_r(w)} |\overline{R}f(z)|\, d\tau(z).$$

In (a) we have proved that the integral I_1 is also at most $C(1 - |w|^2)^{-1/2} \times \int_{E_r(w)} |\overline{R}f(z)|\, d\tau(z)$. This finishes the proof of Lemma 2.1.

Remark. In [12], p. 52, it is shown that $f(w) = \int_S f(z) h(\varphi_z(w))\, d\tau(z)$, where h is a radial function which belongs to $C^\infty(B)$ with compact support in B such that $\int_B h(z)\, d\tau(z) = 1$. Then the argument used in the proof of Lemma 2.1 can be applied to derive the estimate

$$|T_{ij} f(w)| \le C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |f(z)|\, d\tau(z), \qquad w \in B,\ 1 \le i, j \le n.$$

Proof of Theorem 1. In terms of ordinary differential operators the invariant Laplacian $\widetilde{\Delta}$ is as follows:

$$\widetilde{\Delta} = 4(1 - |z|^2) \sum_{j,k=1}^n (\delta_{jk} - z_j \overline{z_k}) \frac{\partial^2}{\partial z_j \partial \overline{z_k}},$$

where δ_{jk} denotes the Kronecker delta; see [12], section 4.1, for details. Using this form for $\widetilde{\Delta}$ and the fact that $\widetilde{\Delta} f = \widetilde{\Delta}\bar{f} = 0$ and $\overline{\partial f/\partial z_j} = \partial \bar{f}/\partial \overline{z_j}$, $1 \le j \le n$, we find that

$$\widetilde{\Delta}|f|^2(z) = 2(1 - |z|^2)|\nabla_T f(z)|^2. \tag{2.1}$$

Also, $|\widetilde{\nabla} f(z)|^2 = 2(|\widetilde{D} f(z)|^2 + |\widetilde{D}\bar{f}(z)|^2) = (1 - |z|^2)|\nabla_T f(z)|^2$ (see [12]). This proves the equivalences of (1), (2), and (3).

An application of the Cauchy-Schwarz inequality shows that

$$|\nabla_T f(z)|^2 \geq 2(1-|z|^2)(|Df(z)|^2 + |D\bar{f}(z)|^2) = (1-|z|^2)|\nabla f(z)|^2.$$

Therefore, (3) implies (4). (We note that quantities $|\nabla f(z)|^2(1-|z|^2)$ and $|\nabla_T f(z)|^2$ are not pointwise equivalent if $n > 1$. If f is a function that depends on one variable only, say z_1, then it is not possible to bound $|\nabla_T f(z)|^2$ by $C(1-|z|^2)|\nabla f(z)|^2$ because $|\nabla_T f(z)|^2 = (1-|z_1|^2)|\nabla f(z)|^2$.)

It is easy to see that (4) implies

$$\sum_{j=1}^n \sup_{z\in B}(1-|z|^2)\left|\frac{\partial f}{\partial z_j}(z)\right| < \infty \quad \text{and} \quad \sum_{j=1}^n \sup_{z\in B}(1-|z|^2)\left|\frac{\partial f}{\partial \overline{z_j}}(z)\right| < \infty,$$

which in turn implies

$$\sup_{z\in B}(1-|z|^2)|Rf(z)| < \infty \quad \text{and} \quad \sup_{z\in B}(1-|z|^2)|\overline{R}f(z)| < \infty.$$

It is easy to check that

$$|z|^2|Df(z)|^2 = |Rf(z)|^2 + \sum_{i<j}|T_{ij}f(z)|^2.$$

Using this, (2.1), and the definition of the tangential gradient we find that

$$\begin{aligned} |z|^2\widetilde{\Delta}|f|^2(z) = 4(1-|z|^2)\Bigg[&(1-|z|^2)(|Rf(z)|^2 + |R\bar{f}(z)|^2) \\ &+ \sum_{i<j}|T_{ij}f(z)|^2 + \sum_{i<j}|T_{ij}\bar{f}(z)|^2\Bigg]. \end{aligned} \tag{2.2}$$

Hence, by (2.1) and (2.2), to show that (5) implies (3) it is sufficient to show that

$$\sum_{i<j}\sup_{z\in B}(1-|z|^2)^{1/2}[|T_{ij}f(z)| + |\overline{T}_{ij}f(z)|] < \infty.$$

An integration by parts shows that

$$f(z) = \int_0^1 [Rf(tz) + \overline{R}f(tz) + f(tz)]\,dt.$$

From this we conclude that it is sufficient to prove that

$$\int_0^1 |T_{ij}u(tz)|\,dt = O\left(\frac{1}{\sqrt{1-|z|^2}}\right), \quad 1 \leq i < j \leq n,$$

where $u(z) = Rf(z)$ or $\overline{R}f(z)$ or $R\bar{f}(z)$ or $\overline{R}\bar{f}(z)$ or $f(z)$.

From

$$f(z) - f(0) = \int_0^1 \frac{d}{dt}f(tz)\,dt = \int_0^1 \frac{1}{t}(Rf(tz) + \overline{R}f(tz))\,dt, \qquad z \in B,$$

we see that $f(z) = O(\frac{1}{1-|z|})$ (in fact, $f(z) = O(\log\frac{1}{1-|z|})$). Thus, $u(z) = O(\frac{1}{1-|z|})$ (note that if $f \in \mathscr{M}$, then $\bar{f} \in \mathscr{M}$ and $|\nabla f(z)| = |\nabla \bar{f}(z)|$, $z \in B$). Using this, Lemma 2.1, the estimate obtained in the remark following Lemma

2.1, the fact that $1-|w|^2 \cong 1-|z|^2$, for $w \in E_r(z)$, and the invariance of measure τ we find that

$$\begin{aligned}\int_0^1 |T_{ij}u(tz)|\,dt &\le C\int_0^1 \left[\frac{1}{(1-t|z|)^{1/2}}\int_{E_r(tz)} |u(w)|\,d\tau(w)\right] dt \\ &\le C\int_0^1 \frac{dt}{(1-t|z|)^{3/2}} \le \frac{C}{(1-|z|)^{1/2}}.\end{aligned}$$

3. Proof of Theorem 3

Lemma 3.1. *Let $k \ge m$ be positive integers, $0<p<\infty$, and $0<r<1$. There exists a constant $C = C(k, m, p, r, n)$ such that if $f \in \mathcal{M}$, then*

$$|\partial^k f(w)|^p \le C(1-|w|)^{(m-k)p}\int_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z), \quad \textit{for all } w \in B.$$

Proof. Let α and β be multi-indices. Using the formula (1.3) in [1] again we find that

$$\begin{aligned}&F(-|\beta|, -|\alpha|, n; r^2)\partial^\alpha\bar\partial^\beta f(w) \\ &\quad= \int_S (1-\langle w, r\xi\rangle)^{-|\alpha|}(1-\langle r\xi, w\rangle)^{-|\beta|}\partial^\alpha\bar\partial^\beta f(r\xi)\,d\sigma(\xi),\end{aligned}$$

where $f(a, b, c; x)$ denotes the usual hypergeometric function. Multiplying this equality by $2nr^{2n-1}(1-r^2)^{-n-1}h(r)\,dr$, where h is a radial function which belongs to $C^\infty(B)$ with compact support in B such that

$$\int_B F(-|\beta|, -|\alpha|, n; |z|^2)h(z)\,d\tau(z) = 1$$

and then integrating from 0 to 1 and using the invariance of the measure τ, we get

(3.1)

$$\begin{aligned}\partial^\alpha\bar\partial^\beta f(w) &= \int_B h(\varphi_w(z))\frac{\partial^\alpha\bar\partial^\beta f(z)\,d\tau(z)}{(1-\langle w, \varphi_w(z)\rangle)^{|\alpha|}(1-\langle\varphi_w(z), w\rangle)^{|\beta|}} \\ &= \int_B h(\varphi_w(z))\frac{(1-\langle w, z\rangle)^{|\alpha|}(1-\langle z, w\rangle)^{|\beta|}}{(1-|w|^2)^{|\alpha|+|\beta|}}\partial^\alpha\bar\partial^\beta f(z)\,d\tau(z),\end{aligned}$$

by Theorem 2.2.2 ([12], p. 26).

Since

$$|1-\langle z, w\rangle| \cong 1-|w|^2, \qquad z \in E_r(w),$$

by a suitable choice of a function h we obtain

$$|\partial^\alpha\bar\partial^\beta f(w)| \le C\int_{E_r(w)} |\partial^\alpha\bar\partial^\beta f(z)|\,d\tau(z).$$

Hence,

$$|\partial^m f(w)| \le C\int_{E_r(w)} |\partial^m f(z)|\,d\tau(z).$$

By Lemma 2.4 ([11]) (see also [2]) we find that

$$|\partial^m f(w)|^p \le C\int_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z).$$

By differentiating under the integral sign in (3.1), using the formula for $\varphi_z(w)$ ([12]), and arguing as above we conclude that

$$|D_j\partial^\alpha\bar\partial^\beta f(w)| \le \frac{C}{1-|w|}\int_{E_r(w)} |\partial^\alpha\bar\partial^\beta f(z)|\,d\tau(z), \qquad w\in B,\ 1\le j\le n,$$

and

$$|\overline{D}_j\partial^\alpha\bar\partial^\beta f(w)| \le \frac{C}{1-|w|}\int_{E_r(w)} |\partial^\alpha\bar\partial^\beta f(z)|\,d\tau(z), \qquad w\in B,\ 1\le j\le n,$$

and so,

$$|\partial^{m+1} f(w)| \le \frac{C}{1-|w|}\int_{E_r(w)} |\partial^m f(z)|\,d\tau(z).$$

By an adaptation of the argument given in ([11], Lemma 2.4) we find that

$$|\partial^{m+1} f(w)|^p \le \frac{C}{(1-|w|)^p}\int_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z).$$

An induction argument shows that

$$|\partial^{k} f(w)|^p \le \frac{C}{(1-|w|)^{(k-m)p}}\int_{E_r(w)} |\partial^m f(z)|^p\,d\tau(z).$$

Proof of Theorem 3. The equivalence of (i) and (ii) is proved in Theorem 1.

If $z\in E_r(w)$, then $1-|w|^2\cong 1-|z|^2$. Hence by Lemma 3.1

$$(1-|z|)^m|\partial^m f(z)| \le C\int_{E_r(z)} (1-|w|)|\partial f(w)|\,d\tau(w) \le C\|f\|_{\mathscr{B}}\tau(E_r(z)),$$

by Theorem 1. Since $\tau(E_r(z)) = r^{2n}(1-r^2)^{-n}$, we have that (ii)$\Rightarrow$(iii).

Conversely, assuming that $\partial^\alpha\bar\partial^\beta f(0)=0$ we have

$$|\partial^\alpha\bar\partial^\beta f(z)| \le \int_0^1\left|\frac{d}{dr}\partial^\alpha\bar\partial^\beta f(rz)\,dr\right| \le C\int_0^1 |\partial^{|\alpha|+|\beta|+1} f(rz)|\,dr.$$

Hence,

$$|\partial^k f(z)| \le C\int_0^1 |\partial^{k+1} f(tz)|\,dt,$$

for any positive integer k. The implication (iii)$\Rightarrow$(ii) follows at once.

Since $\tau(E_r(w))$ is bounded by a constant independent of w, we have that (iii)$\Rightarrow$(iv).

Let $k\ge m$ be a positive integer. Then by Lemma 3.1 we have

$$(1-|z|)^{kp}|\partial^k f(z)|^p \le C\int_{E_r(z)} |\partial^m f(w)|^p(1-|w|)^{mp}\,d\tau(w).$$

Thus, (iv) implies that $\sup_{z\in B}(1-|z|)^k|\partial^k f(z)| < \infty$.

This finishes the proof of Theorem 3.

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