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NON-SMIRNOV DOMAINS

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ABSTRACT. If Ω is a Jordan domain, a small perturbation of the boundary gives a non-Smirnov domain.

Let D be the open unit disc and T its boundary. A conformal mapping $f(z)$ from D onto a Jordan domain Ω extends to a homeomorphism between $\overline{D}$ and $\overline{\Omega}$ (the closures). This was proved by Osgood and Taylor and independently by Carathéodory (1913). The boundary of Ω, $\partial\Omega$, is rectifiable if and only if $f'(z) \in H^1$. See [1]. The H^p spaces are treated in [2] and [4]. The length of $\partial\Omega$ is $|\partial\Omega| = \int_0^{2\pi} |f'(e^{i\theta})|\, d\theta$. Because $f'(z)$ belongs to H^1 we have $f'(z) = S(z)F(z)$, where $S(z)$ is singular inner and $F(z)$ is outer. There is no Blaschke factor since $f(z)$ is univalent. If $S(z) \equiv 1$, then Ω is called a Smirnov domain. In such domains function theory inherits nice properties from the unit disc. See Chapter 10 of [2]. Non-Smirnov domains exist. An elegant proof is due to Duren, Shapiro, and Shields [3]. Keldysh and Lavrentiev gave the first example in 1937. A detailed version appears in the book of Privalov [6].

In this paper we will use the idea of Keldysh and Lavrentiev to show that the shape of such domains can roughly be prescribed. In particular the non-Smirnov domains are dense in the simply connected domains in the sense of Carathéodory.

Theorem. *If $f(z)$ is univalent in D, $0 < r_1 < r_2 < 1$, then there exists a non-Smirnov domain Δ such that $f(|z| < r_1) \subset \Delta \subset f(|z| < r_2)$. There exists a conformal mapping $\varphi(z)$ of D onto Δ such that $|\varphi'(e^{i\theta})| = \text{constant}$ a.e.*

The result has an interesting Brownian motion interpretation. Let I be a measurable subset of $\partial\Delta$. Consider a Brownian motion starting at $\varphi(0)$. The probability for the first exit from Δ to take place on I equals $|I|/|\partial\Delta|$. We need five lemmas.

Lemma 1 (Montel). *Let $\Omega_1 \subset \Omega_2$ be Jordan domains bounded by finitely many analytic arcs. Assume that $f_i(z)$ maps Ω_i conformally onto D and that $f_1(z_0) = f_2(z_0)$ for some $z_0 \in \Omega_1$. If the open analytic arc Γ is contained in $\partial\Omega_1 \cap \partial\Omega_2$, then $|f_2'(z)| \geq |f_1'(z)|$ for every $z \in \Gamma$.*

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A proof is found in [6], p. 28.

Let I be a subarc of T of length $2\theta < \frac{\pi}{3}$. For $0 \le \gamma \le \frac{1}{2}$, C_γ is the part of a circle through the endpoints of I lying outside D making the angle $\gamma\pi$ with I. D_γ is the domain bounded by $(T \backslash I) \cup C_\gamma$. Let $h_\gamma(z)$ map D_γ conformally onto D such that $h_\gamma(0) = 0$. The midpoint of C_γ is z_0.

Lemma 2. *The minimum of $|h'_\gamma(z)|$ for $z \in C_\gamma$ is attained at z_0.*

$$|h'_\gamma(z_0)| = \frac{2}{1+\gamma} \cdot \frac{\tan\frac{\theta}{2(1+\gamma)}\cos^2\frac{\theta+\gamma\pi}{2}}{\sin\theta}$$

A proof is in [6], p. 162.

We want to solve the equation $|h'_\gamma(z_0)| = \mu < 1$ with respect to γ.

Lemma 3. *There exists a constant $c > 0$ such that any solution γ of the equation $|h'_\gamma(z_0)| = \mu < 1$ satisfies $\gamma \ge c(1-\mu)$.*

Proof. By the mean value theorem we have

$$\mu - 1 = |h'_\gamma(z_0)| - |h'_0(z_0)| = \left(\frac{\partial}{\partial\gamma}|h'_\gamma(z_0)|\right)_{\gamma=t} \cdot \gamma$$

for some number t between 0 and γ.

Let $M = \mathrm{Max}\{|\frac{\partial}{\partial\gamma}|h'_\gamma(z_0)||: 0 \le \gamma \le \frac{1}{2}, 2\theta \le \pi/3\}$.

Then $1 - \mu \le M \cdot \gamma$. This proves the lemma with $c = M^{-1}$.

Lemma 4. *There exists a strictly increasing function $g(\gamma)$ satisfying $g(0) = 1$ and $|C_\gamma| \ge g(\gamma)|I|$.*

The proof uses simple geometry and is omitted.

Lemma 5. *Let $\varphi(z)$ be analytic in D. Assume that $|\varphi'(z)| < 1$ and that $\varphi'(0) \ge \delta > 0$. If E is a measurable subset of T of length $s > 0$, then $\int_E |\varphi'(e^{i\theta})|\, d\theta \ge K = K(s, \delta) > 0$.*

Proof. Let $A = \{e^{i\theta}: |\varphi'(e^{i\theta})| < \varepsilon\}$ where ε satisfies $\frac{2\pi\log\delta}{\log\varepsilon} = \frac{s}{2}$. Since $\varphi'(z) \in H^1$, we have $\log\delta \le \frac{1}{2\pi}\int_T \log|\varphi'(e^{i\theta})|\, d\theta \le \frac{1}{2\pi}|A|\log\varepsilon$. Therefore $|A| \le \frac{2\pi\log\delta}{\log\varepsilon} = \frac{s}{2}$. Hence $E(\varepsilon) = \{e^{i\theta} \in E: |\varphi'(e^{i\theta})| \ge \varepsilon\}$ satisfies $|E(\varepsilon)| \ge \frac{s}{2}$. This proves the lemma since $\int_E |\varphi'(e^{i\theta})|\, d\theta \ge \int_{E(\varepsilon)} |\varphi'(e^{i\theta})|\, d\theta \ge \frac{s}{2}\varepsilon = K(s, \delta)$.

We now prove the theorem. The proof is technical and the reader should make a drawing. We will construct Jordan domains Δ_n bounded by a finite number of analytic arcs such that $\Delta_n \subset \Delta_{n+1}$. The non-Smirnov domain will be the union of Δ_n. By $\varphi_n(z)$ we mean the conformal mapping from D onto Δ_n such that $\varphi_n(0) = 0$ and $\varphi'_n(0) = \mathrm{Re}\,\varphi'_n(0) > 0$.

Since Δ_n is bounded by finitely many analytic arcs, $\varphi_n(z)$ has a univalent continuation to a domain D_n containing D. For $n > 1$, ∂D_n meets T at a finite number of points. We define $\varphi_n(D_n) = \Omega_n$. These domains will satisfy $\Omega_n \supset \Omega_{n+1}$. For $n > 1$, $\partial\Omega_n$ will meet $\partial\Delta_n$ at finitely many points. The inverse of $\varphi_n(z)$ is denoted $f_n(z)$. We construct the domains inductively.

We define $\Delta_1 = f(|z| < r_1)$, $D_1 = \{|z| < \frac{r_2}{r_1}\}$ and $\Omega_1 = f(|z| < r_2)$. We may assume that $f(0) = 0$ and that $f'(0) = \mathrm{Re}\, f'(0) > 0$. Then $\varphi_1(z) = f(r_1 z)$ is properly normalized. By dilation we may assume that $|\varphi'_1(z)| \le 1$ for $z \in D$.

Assume that Δ_n, D_n, Ω_n and $\varphi_n(z)$ have been constructed and that $|\varphi_n'(z)| \le 1$ for $z \in D$. Define a_n and A_n by

$$\int_T |\varphi_n'(e^{i\theta})|\, d\theta = |\partial\Delta_n| = 2\pi - a_n,$$

$$A_n = \left\{e^{i\theta} : e^{i\theta} \in D_n,\ |\varphi_n'(e^{i\theta})| < 1 - \frac{a_n}{20}\right\}.$$

This set satisfies $\int_{T\setminus A_n} |\varphi_n'(e^{i\theta})| \le 2\pi - a_n$. Therefore we have $(2\pi - |A_n|) \cdot (1 - \frac{a_n}{20}) \le 2\pi - a_n$. This leads to $|A_n| \ge \frac{a_n}{2}$.

Let $\{I_k\} (= \{I_{k,n}\})$ be finitely many disjoint closed subarcs of A_n. Each I_k has length less than $\frac{\pi}{3}$ and is contained in a closed disc O_k meeting the endpoints of I_k at an angle of $\frac{\pi}{2}$. The following conditions are satisfied:

(i) $O_k \subset D_n$.
(ii) $\sum |I_k| > \frac{a_n}{4}$.
(iii) $\sup_{z\in O_k} |\varphi_n'(z)| = \mu_k < 1 - \frac{a_n}{40}$.
(iv) $\operatorname{diam} O_k < b_n$, where b_n is a small number to be chosen later.
(v) $(\inf_{z\in O_k} |\phi_n'(z)|)/\mu_k > r_n$, where r_n is a number close to one to be chosen later.

Replace I_k by a bubble as in Lemma 2 where γ_k satisfies $|h_{\gamma_k}'(z_0)| = \mu_k$. If no such γ_k exists, let $\gamma_k = \frac{1}{2}$. Recall the definition of D_{γ_k} and let $D_n^* = \bigcup D_{\gamma_k}$. Note that $\varphi_n(z)$ is univalent in D_n^* and that $\varphi_n(D_n^*) \subset \Omega_n$. Let $h_n(z)$ map D_n^* conformally onto D such that $h_n(0) = 0$ and $h_n'(0) > 0$. If $z \in \partial D_n^*$ and $|z| > 1$, then $z \in C_{\gamma_k}$ for some k. Therefore $|h_n'(z)| \ge |h_{\gamma_k}'(z)| \ge |h_{\gamma_k}'(z_0)| \ge \mu_k$. The first inequality follows from Lemma 1, the second from Lemma 2, and the third inequality follows from the the definition of γ_k.

We define $\Delta_{n+1} = \varphi_n(D_n^*)$. Then $\Delta_n \subset \Delta_{n+1} \subset \Omega_n$. The boundary of Δ_{n+1} is rectifiable and consists of a finite number of analytic arcs. To prove that $|\varphi_{n+1}'(z)| \le 1$ for $z \in D$ it suffices to prove that $|f_{n+1}'(z)| \ge 1$ a.e. on $\partial\Delta_{n+1}$. There are two cases. Assume that $z \in \partial\Delta_{n+1} \cap \partial\Delta_n$ and that both $f_n'(z)$ and $f_{n+1}'(z)$ exist. This excludes only a finite number of points on $\partial\Delta_{n+1} \cap \partial\Delta_n$. By Lemma 1 we have that $|f_{n+1}'(z)| \ge |f_n'(z)| \ge 1$. If $z \in \partial\Delta_{n+1} \setminus \partial\Delta_n$, then $f_n(z) \in C_{\gamma_k}$ for some k. Note that $f_{n+1}(z) = h_n(f_n(z))$. Therefore $|f_{n+1}'(z)| = |h_n'(f_n(z))| \cdot |f_n'(z)| \ge \mu_k \cdot \frac{1}{|\varphi_n'(f_n(z))|} \ge \mu_k \cdot \frac{1}{\mu_k} = 1$ by (iii). This proves that $|\varphi_{n+1}'(z)| \le 1$ in D and that $|\partial\Delta_{n+1}| \le 2\pi$. Recall that $\Delta_{n+1} \subset \Omega_n$ and that $\partial\Omega_n$ meets $\partial\Delta_{n+1}$ at a finite number of points. As before $\varphi_{n+1}(z)$ has a univalent continuation to a domain D_{n+1} containing D such that $(\partial D_{n+1} \cap T)$ is finite. If necessary we decrease D_{n+1} by choosing ∂D_{n+1} close to T to obtain:

(1) $\varphi_{n+1}(D_{n+1}) = \Omega_{n+1} \subset \Omega_n$.
(2) Any Jordan curve Γ in $\overline{\Omega}_{n+1}$ surrounding Δ_{n+1} must satisfy $|\Gamma| \ge |\partial\Delta_{n+1}| - \frac{1}{n}$.

We now prove that $|\partial\Delta_n|$ is increasing. It follows from (v) that

$$\frac{\int_{C_{\gamma_k}} |\varphi_n'(z)|\, ds}{\int_{I_k} |\varphi_n'(z)|\, ds} \ge \frac{|C_{\gamma_k}|}{|I_k|} r_n.$$

Note that $\gamma_k \geq c(1-\mu_k) \geq c\frac{a_n}{40}$ by Lemma 3 and (iii). By Lemma 4 $\frac{|C_{\gamma_k}|}{|I_k|} \geq g(c\frac{a_n}{40}) = \zeta' > 1$. Combining these inequalities we obtain

$$\int_{C_{\gamma_k}} |\varphi_n'(z)|\,ds \geq \zeta' r_n \int_{I_k} |\varphi_n'(z)|\,ds.$$

We now choose r_n such that $\zeta' r_n = \frac{1+\zeta'}{2} = \zeta$. Consequently

$$(*)\qquad \begin{aligned} |\partial\Delta_{n+1}| - |\partial\Delta_n| &= \sum_k \left(\int_{C_{\gamma_k}} |\varphi_n'(z)|\,ds - \int_{I_k} |\varphi_n'(z)|\,ds\right) \\ &\geq (\zeta - 1)\sum_k \int_{I_k} |\varphi_n'(z)|\,ds = \frac{g(c\frac{a_n}{40}) - 1}{2} \int_{\cup I_k} |\varphi_n'(z)|\,ds \end{aligned}$$

this proves that $|\partial\Delta_n|$ is increasing.

Since $|\partial\Delta_n| \leq 2\pi$ it follows that $\lim_{n\to\infty} |\partial\Delta_n|$ exists. Assume that this limit equals $2\pi - a$ where $a > 0$. Then $a_n > a$ for all n. Subordination, a variant of Schwarz' lemma, proves that $\varphi_n'(0) \geq \varphi_1'(0) = \delta > 0$. For all n we have that $|\bigcup I_k| > \frac{a}{40}$. Apply Lemma 5 and $(*)$:

$$\begin{aligned} |\partial\Delta_{n+1}| - |\partial\Delta_n| &\geq \frac{g(c\frac{a}{40}) - 1}{2} \int_{\cup I_k} |\varphi_n'(z)|\,ds \\ &\geq \frac{g(c\frac{a}{40}) - 1}{2} K\left(\frac{a}{40}, \delta\right) > 0. \end{aligned}$$

This is a contradiction, hence $\lim |\partial\Delta_n| = 2\pi$. Let $\Delta = \bigcup \Delta_n$. To prove that Δ is a Jordan domain recall that $\Delta_n = \varphi_n(D)$ and that $\Delta_{n+1} = \varphi_n(D_n^*)$. By construction every point of D_n^* can be connected to a point in D by a line segment of length less than b_n. See (iv). Since $|\varphi_n'(z)| < 1$ everywhere in D_n^*, every point in Δ_{n+1} can be connected to a point in Δ_n by a curve of length less than b_n. By induction every point in Δ can be connected to a point in Δ_N by a curve of length less than $\sum_{n\geq N} b_n$. A domain is a Jordan domain if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any two points closer together than δ lie in a connected subset of diameter less than ε. We may assume that $\delta < \varepsilon$. For every N, Δ_N is a Jordan domain. If $\varepsilon_N = \frac{1}{2^N}$ there exists δ_N corresponding to ε_N that works for Δ_N. Choose the numbers b_n in (iv) such that $\sum_{n\geq N} b_n < \frac{1}{3}\delta_N$. Let z_1 and z_2 be two points in Δ such that $|z_1 - z_2| < \frac{\delta_N}{3}$. For $i = 1, 2$ choose curves K_i of length less than $\frac{\delta_N}{3}$ connecting z_i with $w_i \in \Delta_N$. Then $|w_1 - w_2| < \delta_N$. Choose a connected set $E \subset \Delta_N$ of diameter less than ε_N such that $w_i \in E$. The set $(E \cup K_1 \cup K_2)$ is connected, has diameter less than $2\varepsilon_N$, and contains z_i. Hence Δ is a Jordan domain.

Let $\varphi(z)$ be the conformal mapping of D onto Δ normalized by $\varphi(0) = 0$ and $\varphi'(0) = \operatorname{Re}\varphi'(0) > 0$. Since $\varphi_n'(z) \to \varphi'(z)$ uniformly on compact sets, we have that $|\partial\Delta| \leq 2\pi$. Condition (2) shows that $|\partial\Delta| \geq |\partial\Delta_{n+1}| - \frac{1}{n}$. Therefore $|\partial\Delta| = 2\pi$ and the proof is complete unless $\varphi'(z) \equiv 1$. By the dilatation argument in the beginning of the proof we may assume that $\Omega_1 \subset \{|z| < \frac{1}{2}\}$. Since $\Delta \subset \Omega_1$, this cannot be the case.

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