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NEW FORMULAE FOR THE BERNOULLI AND EULER POLYNOMIALS AT RATIONAL ARGUMENTS

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ABSTRACT. We prove theorems on the values of the Bernoulli polynomials $B_n(x)$ with $n = 2, 3, 4, \dots$, and the Euler polynomials $E_n(x)$ with $n = 1, 2, 3, \dots$ for $0 \le x \le 1$ where x is rational. $B_n(x)$ and $E_n(x)$ are expressible in terms of a finite combination of trigonometric functions and the Hurwitz zeta function $\zeta(z, \alpha)$. The well-known argument-addition formulae and reflection property of $B_n(x)$ and $E_n(x)$, extend this result to any rational argument. We also deduce new relations concerning the finite sums of the Hurwitz zeta function and sum some classical trigonometric series.

1. INTRODUCTION

The Bernoulli and Euler polynomials of degree n, denoted respectively by $B_n(x)$ and $E_n(x)$, are defined as [7, p. 25]

$$B_n(x) = \sum_{s=0}^{n} \binom{n}{s} B_s x^{n-s}, \qquad n = 0, 1, 2, \dots, \tag{1a}$$

and [7, p. 39]

$$E_n(x) = \sum_{s=0}^{n} \binom{n}{s} E_s 2^{-s}(x - 1/2)^{n-s}, \qquad n = 0, 1, 2, \dots, \tag{1b}$$

where B_s and E_s are the Bernoulli and the Euler numbers given by the coefficients in the power series

$$\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} B_s \frac{t^s}{s!}, \qquad |t| < 2\pi, \tag{2a}$$

and

$$\frac{1}{\cosh t} = \sum_{s=0}^{\infty} E_s \frac{t^s}{s!}, \qquad |t| < \pi/2. \tag{2b}$$

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The term "Bernoulli polynomials" was introduced by J. L. Raabe in 1851, after Jacob Bernoulli who first studied them (and the "Bernoulli numbers") before 1705. Bernoulli's results were posthumously published in his main work, Ars Conjectandi (1713, p. 97). The Euler polynomials and the Euler numbers (so-named by Scherk in 1825) appear in Euler's famous book, Institutiones Calculi Differentialis (1755, pp. 487–491 and p. 522).

Standard texts on the classical theory of $B_n(x)$ and $E_n(x)$ are Chapter II in Nörlund [9], Chapter VI in Milne-Thomson [8] and Chapters V and VI in Jordan [6]. The differences in definitions and notations in older literature are discussed in [6, p. 230 and p. 290]. A detailed bibliography up to 1960 concerning tables and applications of these polynomials and numbers in summing series, is given in Fletcher et al. [3, pp. 65–117]. An extensive lists of formulae involving $B_n(x)$ and $E_n(x)$ can be found in Erdeley et al. [2, pp. 35–43], Magnus et al. [7, pp. 25–32], Abramowitz and Stegun [1, pp. 803–806], Gradshteyn and Ryzhik [4, pp. 1076–1080] and Prudnikov et al. [10, Vol. 3, pp. 785–766].

Here we are concerned with the values of the Bernoulli and Euler polynomials when x is restricted to the set of rational numbers. Symmetry relations, functional equations and differentiation formulae, representation by trigonometric series and integrals, recurrence formulae and some other properties of these polynomials are mostly last century results. However, it appears that general formulae expressing $B_n(x)$ and $E_n(x)$ at rational arguments in terms of other functions are unknown, since literature lists only several special values of $B_n(x)$ and $E_n(x)$, mostly expressed in terms of the corresponding Bernoulli and Euler numbers (see Section 4 below). Among them is the following celebrated Euler relation (1740)

$$B_{2n}(0) = B_{2n}(1) = B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)$$

between the even-indexed Bernoulli numbers B_{2n} and the values of the Riemann zeta function.

We shall prove the theorems on the values of the Bernoulli polynomials $B_n(x)$ with $n = 2, 3, 4, \ldots,$ and the Euler polynomials $E_n(x)$ with $n = 1, 2, 3, \ldots,$ for $0 \le x \le 1$ where x is rational. Formulae obtained in this way express $B_n(x)$ and $E_n(x)$ in terms of a finite combination of trigonometric functions and the Hurwitz zeta function. Below, we also deduce new relations concerning the Hurwitz zeta function and sum some classical trigonometric series.

2. Statement of results

In what follows $\zeta(z)$ and $\zeta(z, \alpha)$ are respectively the Riemann and the Hurwitz zeta functions defined by [7, p. 19]

$$\begin{aligned}\zeta(z) &= \sum_{k=1}^{\infty}\frac{1}{k^z} = \frac{1}{1-2^{-z}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^z} \qquad (\operatorname{Re} z > 1)\\ &= \frac{1}{1-2^{1-z}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^z} \qquad (\operatorname{Re} z > 0,\ z \neq 1)\end{aligned} \tag{3a}$$

and [7, p. 22]

$$\zeta(z, \alpha) = \sum_{k=0}^{\infty} \frac{1}{(k+\alpha)^z} \qquad (\operatorname{Re} z > 1,\ \alpha \neq 0, -1, -2, \dots). \tag{3b}$$

For $\operatorname{Re} z \leq 1$, $z \neq 1$, the functions $\zeta(z)$ and $\zeta(z, \alpha)$ are defined as the analytic continuations of the foregoing series. Both are analytic over the whole complex plane, except at $z = 1$, where they have a simple pole.

Our main results are as follows.

Theorem A. *Let $B_n(x)$ be the Bernoulli polynomial of degree n, let $\zeta(z, \alpha)$ be the Hurwitz zeta function, and let $x = p/q$ ($p \in Z$, $q \in N$ with $0 \leq p \leq q$). Then:*

$$B_{2n-1}(p/q) = (-1)^n \frac{2(2n-1)!}{(2\pi q)^{2n-1}} \sum_{s=1}^{q} \zeta(2n-1, s/q) \sin(s2\pi p/q), \quad n = 2, 3, 4, \dots, \tag{4a}$$

$$B_{2n}(p/q) = (-1)^{n-1} \frac{2(2n)!}{(2\pi q)^{2n}} \sum_{s=1}^{q} \zeta(2n, s/q) \cos(s2\pi p/q), \quad n = 1, 2, 3, \dots, \tag{4b}$$

Theorem B. *Let $E_n(x)$ be the Euler polynomial of degree n, let $\zeta(z, \alpha)$ be the Hurwitz zeta function, and let $x = p/q$ ($p \in Z$, $q \in N$ with $0 \leq p \leq q$). Then:*

$$E_{2n-1}(p/q) = (-1)^n \frac{4(2n-1)!}{(2\pi q)^{2n}} \sum_{s=1}^{q} \zeta(2n, (2s-1)/2q) \cos((2s-1)\pi p/q), \tag{5a}$$

$$E_{2n}(p/q) = (-1)^n \frac{4(2n)!}{(2\pi q)^{2n+1}} \sum_{s=1}^{q} \zeta(2n+1, (2s-1)/2q) \sin((2s-1)\pi p/q), \tag{5b}$$

where $n = 1, 2, 3, \dots$.

Note 1. The formulae in (4) can be rewritten as

$$B_{2n}(p/q) = (-1)^{n-1} \frac{2(2n)!}{(2\pi q)^{2n}} \left[\sum_{s=1}^{q-1} \zeta(2n, s/q) \cos(s2\pi p/q) + \zeta(2n) \right], \quad n = 1, 2, 3, \dots,$$

since $\zeta(2n, 1) = \zeta(2n)$, and

$$B_{2n-1}(p/q) = (-1)^n \frac{2(2n-1)!}{(2\pi q)^{2n-1}} \sum_{s=1}^{q-1} \zeta(2n-1, s/q) \sin(s2\pi p/q), \quad n = 2, 3, 4, \dots,$$

because one term vanishes. Moreover, it is easy to verify that formulae involved in Theorems A and B can be rewritten as follows:

$$B_n(p/q) = -\frac{2n!}{(2\pi q)^n} \sum_{s=1}^{q} \zeta(n, s/q) \cos(s2\pi p/q - n\pi/2),$$

$$E_n(p/q) = \frac{4n!}{(2\pi q)^{n+1}} \sum_{s=1}^{q} \zeta(n+1, (2s-1)/2q) \sin((2s-1)\pi p/q - n\pi/2),$$

where $p \in Z$, $q \in N$ with $0 \le p \le q$ and $n = 2, 3, 4, \dots$.

Note 2. It is clear that the Theorems A and B could be rewritten in the representation of the polygamma function $\psi^{(n)}$, since there exists the following relationship [1, p. 260, Eq. 6.4.10]

$$\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z), \qquad n = 1, 2, 3, \dots, z \neq 0, -1, -2, \dots,$$

between $\psi^{(n)}$ and the Hurwitz zeta function.

Note 3. Our proof of these theorems, given in §4, involves a Fourier series for the Bernoulli and Euler polynomials and conveniently defined functions based on the Dirichlet power series. The Theorems A and B respectively establish the formulae for the values of the Bernoulli polynomials $B_n(x)$ with $n = 2, 3, 4, \dots$, and the Euler polynomials $E_n(x)$ with $n = 1, 2, 3, \dots$ for $0 \le x \le 1$ where x is rational. In this way their values can be expressed as a finite combination of trigonometric functions and the Hurwitz zeta function. However, by means of the following argument-addition formulae [1, p. 804, entry 23.1.7]

$$B_n(x+m) = \sum_{s=0}^{n} \binom{n}{s} B_s(x) m^{n-s}, \quad E_n(x+m) = \sum_{s=0}^{n} \binom{n}{s} E_s(x) m^{n-s},$$

where m is a nonzero integer, this result may be extended to any rational argument.

3. First results

Let $\Phi(\nu, z)$ denote the Dirichlet power series defined by

$$\Phi(\nu, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} \tag{6}$$

which converges absolutely for all ν if $|z| < 1$, for $\operatorname{Re} \nu > 0$ if $|z| = 1$, $z \neq 1$, and for $\operatorname{Re} \nu > 1$ if $z = 1$. It is known that $\Phi(\nu, z)$ can be extended to the whole ν-plane by means of a contour integral. On setting $z = \exp(i\pi x)$ where x is real, we consider separately the Dirichlet series derived from (6) when n is even and odd.

Lemma. *Let $F_\nu(x)$ and $G_\nu(x)$ be defined by*

$$F_\nu(x) = \sum_{k=1}^{\infty} \frac{\exp\{i2k\pi x\}}{(2k)^\nu}, \quad G_\nu(x) = \sum_{k=0}^{\infty} \frac{\exp\{i(2k+1)\pi x\}}{(2k+1)^\nu} \tag{7}$$

for any real x and each complex ν with $\operatorname{Re} \nu > 1$, and let $\zeta(z, \alpha)$ be defined by (3b). *If x is rational* (*say, $x = p/q$ with $p \in Z$, $q \in N$*), *then*

$$F_\nu(p/q) = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \zeta(\nu, s/q) \exp\{i2s\pi p/q\}, \tag{8a}$$

$$G_\nu(p/q) = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \zeta(\nu, (2s-1)/2q) \exp\{i(2s-1)\pi p/q\}. \tag{8b}$$

Proof. First, note that $F_\nu(x)$ and $G_\nu(x)$ are "well-defined", since the absolute convergence of the corresponding series is assured for any real x when $\operatorname{Re}\nu > 1$. Observe that a shift of index yields

$$F_\nu(p/q) = \sum_{k=1}^{\infty} \frac{\exp\{i2k\pi p/q\}}{(2k)^\nu} = \sum_{k=0}^{\infty} \frac{\exp\{i2(k+1)\pi p/q\}}{(2k+2)^\nu}.$$

Next, recall that for any $a \in Z$, $b \in N$ there exists unique $c, d \in Z$ such that $a = bc + d$ and $0 \le d < b$ (division law in Z). Here, this means that any $(k, q)(k \in N_0, q \in N)$ uniquely determine the integers r and s such that $k = qr + s$ where $r = 0, 1, 2, \ldots$ and $s = 0, 1, \ldots, q-1$. Hence, it follows (by absolute convergence) that

$$\begin{aligned} F_\nu(p/q) &= \frac{1}{2^\nu} \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \frac{\exp\{i2(qr+s+1)\pi p/q\}}{(qr+s+1)^\nu} \\ &= \frac{1}{2^\nu} \sum_{r=0}^{\infty} \sum_{s=1}^{q} \frac{\exp\{i2(qr+s)\pi p/q\}}{(qr+s)^\nu} \\ &= \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \sum_{r=0}^{\infty} \frac{\exp\{i2(qr+s)\pi p/q\}}{(r+s/q)^\nu} \\ &= \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \sum_{r=0}^{\infty} \frac{\exp\{i2r\pi p\} \exp\{i2s\pi p/q\}}{(r+s/q)^\nu} \end{aligned}$$

which can be further simplified to

$$\begin{aligned} F_\nu(p/q) &= \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \sum_{r=0}^{\infty} \frac{\exp\{i2s\pi p/q\}}{(r+s/q)^\nu} \\ &= \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \exp\{i2s\pi p/q\} \sum_{r=0}^{\infty} \frac{1}{(r+s/q)^\nu} \end{aligned}$$

since $\exp\{i2r\pi p\} = 1$ (r and p are integers). In view of the definition of the Hurwitz zeta function in (3b), the proposed formula in (8a) readily follows from the last double sum.

Starting from

$$G_\nu(p/q) = \sum_{k=0}^{\infty} \frac{\exp\{i(2k+1)\pi p/q\}}{(2k+1)^\nu} = \exp(-i\pi p/q) \sum_{k=0}^{\infty} \frac{\exp\{i2(k+1)\pi p/q\}}{(2k+1)^\nu}$$

the formula in (8b) is derived in precisely the same way. This completes the proof of our lemma.

Corollary. *Let* $S_\nu(x)$, $C_\nu(x)$, $T_\nu(x)$, *and* $D_\nu(x)$ *be defined by*

$$\text{(9a)} \qquad \left\{ \begin{matrix} S_\nu(x) \\ C_\nu(x) \end{matrix} \right\} = \sum_{k=1}^{\infty} \frac{1}{(2k)^\nu} \left\{ \begin{matrix} \sin(2k\pi x) \\ \cos(2k\pi x) \end{matrix} \right\},$$

$$\text{(9b)} \qquad \left\{ \begin{matrix} T_\nu(x) \\ D_\nu(x) \end{matrix} \right\} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^\nu} \left\{ \begin{matrix} \sin((2k-1)\pi x) \\ \cos((2k-1)\pi x) \end{matrix} \right\}$$

for any real x and complex ν with $\operatorname{Re}\nu > 1$, and let $\zeta(z, \alpha)$ be defined by (3b). *If x is rational $(x = p/q,\, p \in Z,\, q \in N)$, then*

$$\begin{Bmatrix} S_\nu(p/q) \\ C_\nu(p/q) \end{Bmatrix} = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \zeta(\nu, s/q) \begin{Bmatrix} \sin(2s\pi p/q) \\ \cos(2s\pi p/q) \end{Bmatrix}, \tag{10a}$$

$$\begin{Bmatrix} T_\nu(p/q) \\ D_\nu(p/q) \end{Bmatrix} = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \zeta(\nu, (2s-1)/2q) \begin{Bmatrix} \sin((2s-1)\pi p/q) \\ \cos((2s-1)\pi p/q) \end{Bmatrix}. \tag{10b}$$

Proof. Observe that the definitions of $S_\nu(x)$, $C_\nu(x)$, $T_\nu(x)$, and $D_\nu(x)$ in (9) ensure the convergence of each of the series involved. Combining (7) and (8) and equating the real and imaginary parts on both sides of the expressions in (8) gives the summation formulae in (10).

Note 4. $S_\nu(x)$ and $C_\nu(x)$ are summable for $0 < x \leq 2\pi$ and $\operatorname{Re}\nu > 1$ [10, Vol. 1, p 726, entry 5.4.2.2] in terms of the Hurwitz zeta function, while it appears that such general summation formulae do not exist for $T_\nu(x)$ and $D_\nu(x)$ (see [5, §14.2; 10, Vol. 1, §5.4]). For $n = 1, 2, 3, \ldots$ the sereis $S_{2n-1}(x)$ and $C_{2n}(x)$ [10, Vol. 1, p. 726, entries 5.4.2.5 and 5.4.2.7] as well as $T_{2n-1}(x)$ and $D_{2n}(x)$ [10, Vol. 1, p. 732, entries 5.4.2.3 and 5.4.2.5] are summable in closed form. The former are expressible in terms of the Bernoulli polynomials and the latter in terms of the Euler polynomials (see formulae in (13) and (14) below). It does not seem to have been noticed earlier that the summation formulae in (10) enable a closed-form evaluation of all series in (9) for rational arguments.

Examples. For $\operatorname{Re}\nu > 1$, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^\nu} = \frac{1}{q^\nu} \sum_{s=1}^{q} \zeta(\nu, s/q), \qquad q = 1, 2, 3, \ldots, \tag{11a}$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^\nu} = \frac{1}{q^\nu} \sum_{s=1}^{q} (-1)^{s-1} \zeta(\nu, s/q), \qquad q = 2, 4, 6, \ldots, \tag{11b}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^\nu} = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} \zeta(\nu, (2s-1)/2q), \qquad q = 1, 2, 3, \ldots, \tag{11c}$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^\nu} = \frac{1}{(2q)^\nu} \sum_{s=1}^{q} (-1)^{s-1} \zeta(\nu, (2s-1)/2q), \qquad q = 2, 4, 6, \ldots. \tag{11d}$$

These exaples are the most obvious special cases of the formulae in (10). For instance, by using (9a) and (10a) and letting $p = 0$ and $q = 1, 2, 3, \ldots$ in $C_\nu(x)$, we obtain (11a), a well-known property of the Hurwitz zeta function [5, p. 360, entry 54.13.2]. Similarly, the other sums in (11) can be obtained by a suitable specialization of the formulae in (10). We believe that they are new, since it is clear that the sums

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^\nu} = (1 - 2^{1-\nu}) \zeta(\nu),$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{\nu}} = (1-2^{-\nu})\zeta(\nu),$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{(2k-1)^{\nu}} = 2^{-2\nu}[\zeta(\nu, 1/4) - \zeta(\nu, 3/4)]$$

listed respectively as entries 5.1.2.3, 5.1.4.1, and 5.1.4.2 in [10, Vol. 1], are their special cases.

Further, in view of the definitions of the Riemann zeta function in (3a) and the last sum, the formulae in (11) can be rewritten as follows

$$\sum_{s=1}^{q} \zeta(\nu, s/q) = q^{\nu}\zeta(\nu), \qquad q = 1, 2, 3, \ldots, \tag{12a}$$

$$\sum_{s=1}^{q}(-1)^{s-1}\zeta(\nu, s/q) = q^{\nu}(1-2^{1-\nu})\zeta(\nu), \qquad q = 2, 4, 6, \ldots, \tag{12b}$$

$$\sum_{s=1}^{q} \zeta(\nu, (2s-1)/2q) = (2q)^{\nu}(1-2^{-\nu})\zeta(\nu), \qquad q = 1, 2, 3, \ldots, \tag{12c}$$

(12d)

$$\sum_{s=1}^{q}(-1)^{s-1}\zeta(\nu, (2s-1)/2q) = (2q)^{\nu}\beta(\nu) = 2^{-\nu}q^{\nu}[\zeta(\nu, 1/4) - \zeta(\nu, 3/4)],$$

$$q = 2, 4, 6, \ldots,$$

where β in (12d) denotes the series on left-hand side of (11d). As was already mentioned, the relation in (12a) is known, but we believe that the others are new.

4. Proof of Theorems A and B

Proof of Theorem A. The Bernoulli polynomials $B_n(x)$ are represented by the following the Fourier series [1, p. 805, entry 23.1.17 and 23.1.18]

$$B_{2n-1}(x) = (-1)^{n}\frac{2(2n-1)!}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{k^{2n-1}} \tag{13a}$$

where $0 \le x \le 1$ for $n = 2, 3, \ldots$, $0 < x < 1$ for $n = 1$, and

$$B_{2n}(x) = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2k\pi x)}{k^{2n}} \tag{13b}$$

where $0 \le x \le 1$ for $n = 1, 2, 3, \ldots$. By combining (9a), (10a), and (13), we arrive at the formulae in (4), which completes the proof.

Proof of Theorem B. The Euler polynomials $E_n(x)$ of degree n are represented by the following Fourier series [1, p. 805, entry 23.1.17 and 23.1.18]

$$E_{2n-1}(x) = (-1)^{n}\frac{4(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos((2k+1)\pi x)}{(2k+1)^{2n}} \tag{14a}$$

where $0 \le x \le 1$ for $n = 1, 2, 3, \dots$, and

$$E_{2n}(x) = (-1)^n \frac{4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{(2k+1)^{2n+1}} \tag{14b}$$

where $0 \le x \le 1$ for $n = 1, 2, 3, \dots, 0 < x < 1$ for $n = 0$. By combining (9b), (10b), and (14), we arrive at the formula in (5), which completes the proof.

In order to relate our results to the already known special values of $B_n(x)$ and $E_n(x)$ we consider the following examples.

Examples. Let B_n be nth Bernoulli number. Then for $n = 1, 2, 3, \dots$. we have

(a) $B_{2n}(0) = B_{2n}(1) = B_{2n} = (-1)^{n-1} \dfrac{2(2n)!}{(2\pi)^{2n}} \zeta(2n)$;

(b) $B_{2n+1}(0) = B_{2n+1}(1) = 0$,

$B_{2n}(1/2) = (1/2)(2^{1-2n} - 1)B_{2n}$,

$B_{2n}(1/3) = B_{2n}(2/3) = (1/2)(3^{1-2n} - 1)B_{2n}$,

$B_{2n}(1/6) = B_{2n}(5/6) = (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}$,

$E_{2n-1}(0) = -E_{2n-1}(1) = -(1/n)(2^{2n} - 1)B_{2n}$,

$E_{2n-1}(1/3) = -E_{2n-1}(2/3) = (1/2n)(2^{2n} - 1)(3^{1-2n} - 1)B_{2n}$.

Proof. These examples give the known special values of the Bernoulli and Euler polynomials [10, Vol. 3, pp. 765–766]. Here, we shall prove the above formulae without appealing to the theory of $B_n(x)$ and $E_n(x)$, and using only our Theorems A and B and the property of the Hurwitz zeta function given in (12a).

(a) This is the Euler relation. We are not aware of a shorter proof (see, for instance, review in [11]), since it is obviously a special case of the formula in (4b) and readily follows form it on putting $p = 0, q = 1$, or $p = 1, q = 1$.

(b) First, we derive the relations

$$\begin{aligned} &\zeta(\nu, 1/2) = (2^\nu - 1)\zeta(\nu), \\ &\zeta(\nu, 1/3) + \zeta(\nu, 2/3) = (3^\nu - 1)\zeta(\nu), \\ &\zeta(\nu, 1/6) + \zeta(\nu, 5/6) = (2^\nu - 1)(3^\nu - 1)\zeta(\nu) \end{aligned} \tag{15}$$

by making use of the property of the Hurwitz zeta function given in (12a). All the above formulae for special values can then be derived by applying the Euler relation and formulae in (15). As an illustration, we calculate the value of $B_{2n}(1/6)$. By making use of (4b) and putting $p = 1$ and $q = 6$ for $n = 1, 2, 3, \dots$ we have

$$\begin{aligned} B_{2n}(1/6) &= (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{2^{2n} 3^{2n}} \{(1/2)[\zeta(2n, 1/6) + \zeta(2n, 5/6)] \\ &\quad - (1/2)[\zeta(2n, 1/3) + \zeta(2n, 2/3)] + [\zeta(2n, 1) - \zeta(2n, 1/2)]\} \\ &= (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{2^{2n} 3^{2n}} (1/2)(2^{2n} - 2)(3^{2n} - 3)\zeta(2n) \\ &= (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)(-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \\ &= (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}. \end{aligned}$$

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NOTE ADDED IN PROOF

After submitting the manuscript we became aware that our formula (4a) had been derived in a completely different way by G. Almkvist and A. Meurman, C. R. Math. Rep. Acad. Canada **13** (1991), 104. Also, a complete bibliography on Bernoulli and Euler polynomials can be found in K. Dilcher, L. Skula, and I. Sh. Slavutskii, *Bernoulli numbers*, 1713/1990, Queen's Papers in Pure and Appl. Math. **87** (1990).

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