FREE AMALGAMS OF RANK TWO

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ABSTRACT. For every $n > 1$ and $m > 1$ we construct an amalgamated free product $U *_W V$ which is free of rank 2, while U is (free) of rank n , V is of rank m , and W is, necessarily, of rank $n + m - 2$.

1. INTRODUCTION

If U and V are groups, then a “common” subgroup is a group W and two monomorphisms $i_1 : W \rightarrow U$ and $i_2 : W \rightarrow V$. Given a monomorphism $i : W \rightarrow U$ we say that W is a free factor of U if $U = i(W) * W'$, i.e., there is a subgroup W' of U such that $i(W)$ and W' generate U and the natural map from $W * W'$ to U is an isomorphism. When U is free this is the same as saying that, for every basis of W , its image in U extends to a basis of U .

It is clear that if U and V are free and W is a common free factor, then $U *_W V$ is a free group. Are there other cases in which $U *_W V$ is free? The following example, which I first learned from Amnon Rosenmann, shows that there are.

Let G be the group given by the presentation

$$\langle a, b, x, y : y = a^2, b = x^2 \rangle.$$

It is clear that G is free of rank 2, a basis being a, x . Take U to be the free group on a, b ; V the free group on x, y ; and W the free group on s, t . Let $i_1 : W \rightarrow U$ be defined by $i_1(s) = a^2$, $i_1(t) = b$, and $i_2 : W \rightarrow V$ by $i_2(s) = y$, $i_2(t) = x^2$. It is evident that W is not a free factor in either U or V and it is easily seen that

$$G \approx U *_W V.$$

In this example, as in the examples where W is a free factor of U or V , the ranks satisfy the formula

$$(*) \quad \text{rank}(G) = \text{rank}(U) + \text{rank}(V) - \text{rank}(W).$$

In fact this formula is true generally: if U and V are free groups of finite rank and $G = U *_W V$ is free, then $(*)$ holds. The proof follows from considerations

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of Euler characteristic. Recall that if F is a free group of rank n its Euler characteristic, $\chi(F)$, is $1 - n$ (see, e.g., [1, p. 247]) and that

$$\chi(U *_W V) = \chi(U) + \chi(V) - \chi(W)$$

([1, p. 250]). Thus

$$1 - \text{rank}(G) = 1 - \text{rank}(U) + 1 - \text{rank}(V) - (1 - \text{rank}(W))$$

from which (*) follows.

Assuming that W is a proper subgroup in both U and V (i.e., that $i_1(W) \neq U$ and $i_2(W) \neq V$), then $G = U *_W V$ is certainly noncommutative and thus, if it is free, its rank is at least 2. We see that when W is a proper subgroup in U and V a necessary condition for G to be free is

$$\text{rank}(W) \leq \text{rank}(U) + \text{rank}(V) - 2.$$

In this note we construct, for every $n > 1$ and $m > 1$, a free amalgam $U *_W V$ in which

$$\text{rank}(U) = n, \quad \text{rank}(V) = m, \quad \text{rank}(W) = n + m - 2,$$

i.e.,

$$\text{rank}(U *_W V) = 2.$$

2. A TECHNICAL LEMMA

Our main tool will be the following technical

(2.1) **Lemma.** *If $a_1, \dots, a_n$ is a basis of a free group, then, for every $k \geq 0$, there is a subgroup of rank $n + k$ that has a basis of the form*

$$a_1^2, a_2, b_1, \dots, b_k, a_3, \dots, a_n,$$

with each b_i a word in a_1 and a_2 , i.e., $b_i \in \langle a_1, a_2 \rangle$ for $i = 1, \dots, k$.

The proof of this lemma is an application of a remarkable theorem of Hall and Rado [3]. To explain it we need some terminology and notation. Let

$$\mathcal{B} = \{a_1, \dots, a_n\}$$

be a basis of a free group F , as in the lemma. A *Schreier set* (or *Schreier tree*), with respect to this basis, is a nonempty set, P , of reduced words such that if $c_1 \cdots c_r$, with each c_i or c_i^{-1} in $\mathcal{B}$, is a reduced word in P , then $c_1 \cdots c_{r-1}$ also belongs to P . In particular, 1_F , which is the empty word, is always in P . Schreier showed that every subgroup of F has a Schreier set of representatives of the right cosets and that this set has remarkable properties. Hall and Rado posed the converse question: if P is a Schreier set, is it the set of representatives of some subgroup? They answered it by proving the following. For each a_i ($i = 1, \dots, n$) let P^{i+} be defined by

$$P^{i+} = \{c \in P : ca_i \notin P\}$$

and let

$$P^{i-} = \{c \in P : c \notin Pa_i\}.$$

Graphically we can view each a_i as effecting a “flow” on P , by moving each of its elements c to ca_i . Some elements of P are, thereby, pushed out of P and

this is P^{i+} . Thus the elements of P^{i+} will be called also “exits” in direction i . Similarly some positions in P become vacant after the transformation, and this is P^{i-} . An element of P^{i-} will be called an “entrance” in direction i . Hall and Rado proved that a necessary and sufficient condition that P be a set of representatives of a subgroup is that the following set of conditions (the “HR conditions”) should hold: for each i , $i = 1, \dots, n$, the sets P^{i+} and P^{i-} are in bijection. Moreover if for each i , φ_i is a bijection from P^{i+} to P^{i-} , then the set

$$\{ca_i\varphi_i(c)^{-1} : c \in P^{i+}, i = 1, \dots, n\}$$

is the basis of a (free) subgroup that has P as a set of representatives of its right cosets.

We apply the HR-theorem as follows. In the notation of our lemma let H be the subgroup generated by $a_1^2, a_2, \dots, a_n$. It is easy to see (e.g., using Hall’s theorem) that a Schreier tree of representatives of the right cosets of H is the set

$$P = \{1, a_1\} \cup P'$$

where P' is the set of all words that, in reduced form, start with a a_1a_i or $a_1a_i^{-1}$ for some $i = 2, \dots, n$. Let R be the subtree of P that contains only words in a_1 and a_2 (and their inverses). It is clear that we can find k positions (i.e., elements)

$$w_1, \dots, w_k$$

in R that have the properties: (i) none is a prefix of any of the other elements (a group element u is a prefix of an element v if u is an initial segment of v when v is written in reduced form). (ii) Each of them, when written in reduced form, ends with a_1a_2 . So each w_i is a word in $a_1^{\pm 1}, a_2^{\pm 1}$ and has a reduced expression

$$w_i = u_ia_2.$$

We construct a new Schreier tree, T , by erasing from P all the offsprings of u_i , for every $i = 1, \dots, k$, in the a_2 and the a_2^{-1} directions. In other words, we erase all words in P for which some w_i or $u_ia_2^{-1}$ is a prefix, including the words w_i and $u_ia_2^{-1}$ themselves. It is clear that T is indeed a Schreier tree.

I claim that T satisfies the HR-conditions. The original tree, P , has one exit and one entrance in each direction. In direction a_1 its exit is a_1 and its entrance is 1_F . In any other direction 1_F is its only exit and entrance. It is clear that in T we have added exits only in the a_2 direction. (Note that exits in “negative” directions, such as a_2^{-1} , do not count; this is because the Hall-Rado theorem tells us that we only need worry about the positive directions. The negative directions take care of themselves.) Each new exit is some u_i and, as we erased from P the descendents of all u_i ’s in the a_2^{-1} direction, it is also an entrance.

It is also easy to see that the u_i ’s are the only new entrances in T . Indeed if e is a new entrance, then this means that e is not in Ta_i for some i but, being new, it is in Pa_i . Thus e has to have the form $e'a_i$ for some e' which is in P and not in T . This means that e' is a near neighbour of $w_j = u_ja_2$ or $u_ja_2^{-1}$ for some $j = 1, \dots, k$. But it is clear now that, if $e'a_i$ is to be in T , e' must be equal to $u_ja_2^{-1}$ and $i = 2$ so that $e'a_i = u_j$. Thus we see that

the set

$$\{u_1, \dots, u_k\},$$

which is the set of all new exits, is also the set of all new entrances. Hence we can make correspond, to each new exit, itself—considered as a new entrance. This correspondence extends the “old” correspondence that gave H . The new, extended, correspondence gives a group whose basis contains the basis $\{a_1^2, a_2, \dots, a_n\}$ of H and, in addition, contains the k elements

$$u_i a_2 u_i^{-1}, \quad i = 1, \dots, k.$$

These k elements are, evidently, words in $a_1^{\pm 1}, a_2^{\pm 1}$ only.

Altogether this new group satisfies all our requirements, thereby proving the lemma.

3. AMALGAMS OF RANK TWO

In this section we prove

(3.1) **Theorem.** *For every $n > 1$ and $m > 1$ there is an amalgam $U *_W V$ which is free of rank two, where U is free of rank n , and V is free of rank m .*

The idea is to compress “most” of U into a small portion of V and most of V into a small portion of U , without getting into an infinite vicious circle. Avoiding a vicious circle is through the a_1^2 in the lemma.

Let U be the free group on a basis $a_1, \dots, a_n$ and let W_1 be a subgroup of rank $n + m - 2$ as in the lemma. In other words, W_1 has a basis

$$a_1^2, a_2, b_1, \dots, b_{m-2}, a_3, \dots, a_n$$

where $b_i \in \langle a_1, a_2 \rangle$ for $i = 1, \dots, m - 2$. Similarly let V be a free group on a basis $x_1, \dots, x_m$ and let W_2 be a subgroup of rank $n + m - 2$ that has a basis

$$y_1, \dots, y_{n-1}, x_2, x_3, \dots, x_m$$

where $y_1, \dots, y_{n-1} \in \langle x_1, x_2 \rangle$. Such subgroups exist, e.g., by the lemma.

We will amalgamate U and V by decreeing that

$$x_2 = a_1^2, \quad x_3 = b_1, \dots, \quad x_m = b_{m-2}$$

and also

$$a_2 = y_1, \quad a_3 = y_2, \dots, \quad a_n = y_{n-1}.$$

The resulting group will be called G . It can also be described as follows. Let W be a free group on a basis $w_1, \dots, w_{n+m-2}$. Map W to V by

$$\begin{aligned} w_1 &\rightarrow x_2, \quad w_2 \rightarrow x_3, \dots, \quad w_{m-1} \rightarrow x_m, \\ w_m &\rightarrow y_1, \quad w_{m+1} \rightarrow y_2, \dots, \quad w_{n+m-2} \rightarrow y_{n-1}. \end{aligned}$$

Similarly map W to U by

$$\begin{aligned} w_1 &\rightarrow a_1^2, \quad w_2 \rightarrow b_1, \dots, \quad w_{m-1} \rightarrow b_{m-2}, \\ w_m &\rightarrow a_2, \quad w_{m+1} \rightarrow a_3, \dots, \quad w_{n+m-2} \rightarrow a_n. \end{aligned}$$

Then it is clear that G is isomorphic to $U *_W V$ by an obvious isomorphism, and we will identify the two groups.

We claim that G is generated by a_1 and x_1 . To show this it will suffice to show that every element of U and every element of V is generated by a_1 and x_1 . Let $u \in U$. Then u is a word in $a_1, \dots, a_n$, $u = u(a_1, \dots, a_n)$. Substituting y_1 for a_2 , y_2 for a_3 , etc., we get

$$u = u(a_1, y_1, y_2, \dots, y_{m-1}).$$

Each y_i is a word in x_1, x_2 ; and x_2 , in turn, is equal to a_1^2 . So u is a word in a_1 and x_1 . A similar proof works for elements of V .

It remains to prove that a_1, x_1 generate G freely. For that we will use the following principle, which was already used implicitly in Rosenmann's example mentioned above. Suppose that in a group presentation

$$\langle c_1, \dots, c_p : r_1, \dots, r_q \rangle$$

there is a relation of the form

$$c_j = w(c_1, \dots, c_{j-1}, c_{j+1}, \dots),$$

expressing a generator as a word in the other generators. Then one can delete that generator and that relation, substitute w for c_j in all other relations in which it occurs, and these changes do not effect the group being presented. In our case one can clearly delete $x_2, x_3, \dots, x_m$ while substituting a_1^2 for x_2 and b_i for x_{i+2} for $i = 1, \dots, m-2$. One can, then, delete $a_2, \dots, a_n$ and the relations in which they occur, thereby remaining with no relators. Thus G is free on a_1, x_1 , as claimed.

4. ADDITIONAL REMARKS

We have seen that the condition

$$\text{rank}(W) \leq \text{rank}(U) + \text{rank}(V) - 2$$

is necessary for $U *_W V$ to be free and the theorem shows that the inequality is "sharp".

There is another necessary condition for $U *_W V$ to be free. To explain it we will introduce some terminology. If a group X is a subgroup of finite index in a group Y we will say that Y fi-contains X . It is clear that if W is a common subgroup of U and of V , U' is a subgroup of U that properly fi-contains (the image of) W and V' is a subgroup of V that properly fi-contains W , then $U *_W V$ contains

$$H = U' *_W V'$$

and H is not of finite index over W . However, a well-known result of Greenberg (see [2]) says that in free group if two subgroups fi-contain another subgroup their join is also of finite index over that subgroup. This shows that if such U' and V' as above exist, then $U *_W V$ cannot be free. In fact, a finitely generated subgroup, X , of a free group always has a "root", meaning a subgroup that fi-contains it and contains every subgroup that fi-contains it (see [6]); we denote it by $\text{root}(X)$. If $\text{root}(X) = X$ we say that X is "root-closed". We can now say that a necessary condition that $U *_W V$ be free is that (the image of) W be root-closed in at least one of the factors U, V .

It is not true, however, that if W is root-closed in both U and V and it satisfies the rank inequality (*), then $U *_W V$ is free. An example is the amalgam

$$\langle a, b, x, y : a^2 b^2 = x^2 y^2 \rangle.$$

The element a^2b^2 is not a square in the free group $\langle a, b \rangle$: this is a theorem of Lyndon [4]. By going to the abelianization we see that it is not a p th power for every odd prime p . Thus the cyclic group it generates is root-closed in $\langle a, b \rangle$. Similarly the cyclic group $\langle x^2y^2 \rangle$ is root-closed in the free group $\langle x, y \rangle$. But the amalgam is not free. To see this note that the amalgam is a 1-relator group, the relator being $a^2b^2y^{-2}x^{-2}$. By a consequence of the freiheitsatz (originally due to Whitehead) if a one-relator group $\langle c_1, \dots, c_n : r \rangle$ is free, then r is in some basis (see [5, p. 106]). But it is clear, again by looking at the abelianization, that $a^2b^2y^{-2}x^{-2}$ is not in any basis of the free group $\langle a, b, x, y \rangle$.

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