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THE DIAMETER CONJECTURE FOR QUASICONFORMAL MAPS IS TRUE IN SPACE

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ABSTRACT. The diameter conjecture for quasiconformal maps is a natural generalization of the Hayman-Wu theorem on level sets of a univalent function. Astala, Fernández, and Rohde recently disproved this conjecture in the plane. Here we show it is true in space.

1. INTRODUCTION

Suppose that f is a K-quasiconformal map of a domain D in $\mathbf{R}^n$ onto the unit ball $\mathbf{B}^n$, and suppose that L is a line in $\mathbf{R}^n$, $n \geq 2$. The *diameter conjecture* states

$$\sum_i (\operatorname{diam} fL_i)^{n-1} \leq C(n, K) < \infty, \tag{1.1}$$

where we sum over the components L_i of $L \cap D$. This undeniably esoteric-looking conjecture originates in the celebrated theorem of Hayman and Wu [HaW], asserting that the length of $f(L \cap D)$ is bounded by an absolute constant whenever $n = 2$ and f is conformal. It was shown in [FHM] that for conformal maps the length of fL_i is comparable to its diameter, and (1.1) was conjectured there for quasiconformal maps in the plane. The n-dimensional version of the conjecture was stated by Väisälä [V2], who also proved that the sum in (1.1) converges for all $n \geq 2$, provided the power $n - 1$ is replaced by any $p > n - 1$, and diverges in general for powers $p < n - 1$. Subsequently, Astala, Fernández, and Rohde [AFR] constructed a counterexample disproving the diameter conjecture in the plane.

In this note we prove:

1.2. **Theorem.** *The diameter conjecture is true for* $n \geq 3$.

There is one definite reason for this dimensional break: the conjecture is true in space because quasiconformal balls are subject to more severe restrictions there than in the plane. More precisely, we shall make decisive use of the

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fact that the complement of a quasiconformal ball in space is linearly locally connected (see Fact 3 below).

There is yet another way to view the Hayman-Wu theorem in the plane: if (x_j) is a hyperbolically separated sequence on $L \cap D$, then $(f(x_j))$ is an interpolating sequence for bounded analytic functions in the unit disk or, equivalently, $\sum_j (1-|f(x_j)|)\delta_{f(x_j)}$ is a Carleson measure in $\mathbf{B}^2$. This was observed in [GGJ]. Similar result can be formulated and proved in higher dimensions as well. We say that a sequence (x_i) in D is *separated* with constant $\eta > 0$ if

$$B(x_i, \eta \operatorname{dist}(x_i, \partial D)) \cap B(x_j, \eta \operatorname{dist}(x_j, \partial D)) = \varnothing$$

for all $i \neq j$. Here and throughout the paper $B(x, r)$ denotes an open n-ball centered at x with radius r.

1.3. **Theorem.** *If $n \geq 3$ and (x_i) is a separated sequence on $L \cap D$ with constant η, then*

$$\sum_{f(x_i) \in B} (1 - |f(x_i)|)^{n-1} \leq C(n, K, \eta)(\operatorname{diam} B)^{n-1} \tag{1.4}$$

for all balls B centered at $\partial \mathbf{B}^n$.

By [AFR], also Theorem 1.3 fails for $n = 2$. In [V2] Väisälä considered a more general case where L is replaced with a curve satisfying a "three point condition". With appropriate modifications, our argument can be used to establish Theorems 1.2 and 1.3 for such curves as well; for notational simplicity, we forgo this general situation. We do not know whether one can replace L in Theorem 1.3 with a p-dimensional plane for $2 \leq p < n$.

Theorem 1.2 is proved in §§2 and 3, and an outline for the proof of Theorem 1.3 is given in §4. In §5 we collect some folklore about quasihyperbolic geodesics, needed in the proof of the diameter conjecture.

By the measurable Riemann mapping theorem, in the plane (1.1) is equivalent to the case when L is a (K-)quasicircle and f is conformal. The counterexample in [AFR] is rather complicated, and one can show that in a sense any such counterexample must be complicated. It is still an open problem to characterize the situations when (1.1) holds in the plane; some sufficient and necessary criteria are given in [AFR] and [HeW].

Articles [GGJ], [FHM], and [Ø] offer various proofs of the Hayman-Wu theorem, and extensions for other types of curves appear in [FH] and [BJ]. A different generalization of the Hayman-Wu theorem to higher dimensions was proved by Wu in [W].

2. Preliminary reductions

We equip L with a natural ordering and then assume that each component L_i is a bounded interval (a_i, b_i) with end points $a_i < b_i$. An easy argument shows that $\lim f(x)$ exists when x approaches either of the end points a_i or b_i along L_i (see [FHM, p. 126]); the respective limits will be denoted by $f(a_i)$ and $f(b_i)$. Let c_i be a point on L_i such that

$$\operatorname{diam} f[a_i, c_i] = \operatorname{diam} f[c_i, b_i]. \tag{2.1}$$

Then by [V2, Theorem 2.3] there is a constant $C_0 = C_0(n, K)$ such that both arcs $f[a_i, c_i]$ and $f[c_i, b_i]$ are of *C_0-bounded turning*; this means that the diameter of any subarc is less than C_0 times the distance between its end points.

Next, let m_i be the midpoint of L_i, and let $z_i = m_i$, if $m_i < c_i$, and $z_i = c_i$ otherwise. Then we have by (2.1) that

$$\operatorname{diam} fL_i \le \operatorname{diam} f[a_i, c_i] + \operatorname{diam} f[c_i, b_i] = 2 \operatorname{diam} f[a_i, c_i].$$

In particular, if $z_i = c_i \le m_i$, then $\operatorname{diam} fL_i \le 2 \operatorname{diam} f[a_i, z_i]$, and it follows, by symmetry, that we only need to verify

$$\sum_i (\operatorname{diam} f[a_i, z_i])^{n-1} \le C(n, K) < \infty. \tag{2.2}$$

To reduce the situation further, let γ_i be a quasihyperbolic geodesic from z_i to a_i in D, as described in §5. Then by the bounded turning property of $f[a_i, z_i]$ and by [HN, Theorem 6.1] there is a point $x_i \in \gamma_i$ such that

$$\operatorname{diam} f[a_i, z_i] \le C_0 |f(a_i) - f(z_i)| \approx \operatorname{diam} f\gamma_i \approx 1 - |f(x_i)|. \tag{2.3}$$

Here and in what follows we use the notation $A \approx B$ to indicate that $C^{-1}A \le B \le CA$ for some constant $C > 0$ depending only on n and K.

We deduce from (2.2) and (2.3) that it suffices to prove

$$\sum_i (1 - |f(x_i)|)^{n-1} \le C(n, K) < \infty, \tag{2.4}$$

where x_i is as chosen above. In the course of the proof we shall need to replace some of the points x_i with new points x_i' that do not necessarily lie on γ_i but satisfy

$$1 - |f(x_i)| \le C(n, K)(1 - |f(x_i')|).$$

To prove (2.4) we need the following three facts.

2.5. **Fact 1** [HK, Lemma 6.6]. *If $y = f(x)$ is a point in $\mathbf{B}^n$, there is a constant $C_1 = C_1(n, K)$ and a set $S = S_y$ on the boundary $\partial\mathbf{B}^n$ such that*

$$(1 - |y|)^{n-1} \le C_1 |S| \tag{2.6}$$

and

$$f^{-1}(w) \in B(x, C_1 \operatorname{dist}(x, \partial D)) \tag{2.7}$$

for all $w \in S$, where $|S|$ denotes the $(n-1)$-measure of S. Here we think of $f^{-1}|\partial\mathbf{B}^n$ as its radial extension which is defined a.e. (in fact, capacity everywhere) on $\partial\mathbf{B}^n$.

We see from (2.6) that to prove (2.4) it suffices to show that

$$\sum_i \chi_i(w) \le C(n, K) < \infty$$

for all $w \in \partial\mathbf{B}^n$, where χ_i is the characteristic function of $S_{f(x_i)}$. Furthermore, by (2.7) it suffices to show that each $x \in \mathbf{R}^n$ belongs to at most $C(n, K)$ balls $B(x_i, C_1 \operatorname{dist}(x_i, \partial D))$.

2.8. **Fact 2** [HN, Theorem 6.2]. *There is a constant $C_2 = C_2(n, K)$ such that*

$$\operatorname{diam} \gamma_i \le C_2 |a_i - z_i|.$$

2.9. **Fact 3** [G, Lemma 1]. *The complement $\complement D$ of D in $\mathbf{R}^n \cup \{\infty\}$ is C_3-linearly locally connected with $C_3 = C_3(n, K)$. This means that for each $x \in \mathbf{R}^n$*

and $R > 0$ *each pair of points in* $\overline{B}(x, R) \cap \complement D$ *can be joined in* $\overline{B}(x, RC_3) \cap \complement D$ *and each pair of points in* $\complement D \setminus B(x, R)$ *can be joined in* $\complement D \setminus B(x, R/C_3)$.

Furthermore, if $a, b \in \partial B(x, R) \cap \complement D$, then they belong to the same component of $(\overline{B}(x, C_3 R) \setminus B(x, R/C_3)) \cap \complement D$. This can be verified by the method used in [GV, Theorem 6.1] and [G, Lemma 1].

3. Proof of Theorem 1.2

As seen above, it suffices to show that each $x \in \mathbf{R}^n$ belongs to at most $C(n, K)$ balls $B_i = B(x_i, C_1 \operatorname{dist}(x_i, \partial D))$. The points x_i are defined as in (2.3) but, as alluded to in §2, are subject to change.

We define generations $\mathscr{L}_\nu$ for $\nu \in \mathbf{Z}$ as

$$L_i \in \mathscr{L}_\nu \quad \text{if and only if} \quad 2^{-\nu-1} < |a_i - x_i| \leq 2^{-\nu}.$$

We may then assume that

$$L_i \in \mathscr{L}_{\nu_i} \text{ and } L_j \in \mathscr{L}_{\nu_j} \quad \text{implies} \quad \nu_i = \nu_j \text{ or } |\nu_i - \nu_j| \geq N, \tag{3.1}$$

where N is a large number, adjusted later, depending only on n and K.

Fix i and write $R_i = |a_i - x_i|$. The line L meets the sphere $\partial B(a_i, R_i)$ at two antipodal points $a_i' < a_i < b_i'$. Using Fact 3, it is easily seen that there is a point

$$w_i \in \complement D \cap (\partial B(a_i, R_i) \setminus B(a_i', R_i/C_4) \setminus B(b_i', R_i/C_4))$$

for some $C_4 = C_4(n, K)$. More precisely, first join ∞ to a_i by a continuum F in $\complement D \setminus B(a_i', R_i/C_3)$. If F meets $B(b_i', R_i/C_3^3)$, we can pick points $w_{a_i} \in \partial B(b_i', R_i/C_3^2) \cap F_{a_i}$ and $w_\infty \in \partial B(b_i', R_i/C_3^2) \cap F_\infty$, where F_{a_i} and F_∞ are, respectively, the a_i and ∞ components of $F \cap \complement B(b_i', R_i/C_3^2)$. By the last assertion in Fact 3, we can join w_{a_i} to w_∞ by a continuum F' in $\overline{B}(b_i', R_i/C_3) \setminus B(b_i', R_i/C_3^3)$. Thus the desired point w_i can be found from the set $F \cup F'$, and we can choose $C_4 = C_3^3$. It is exactly here where the argument would fail in two dimensions: no such w_i need exist.

Write $d_i = \operatorname{dist}(x_i, \partial D)$ and let $\Delta_i = B(x_i, d_i/2)$. Standard distortion estimates for quasiconformal maps (see [V1, 18.1]) together with (2.3) imply

$$\operatorname{diam} f\Delta_i \approx 1 - |f(x_i)| \approx \operatorname{diam} f\gamma_i \geq \operatorname{dist}(f\Delta_i, f[a_i, z_i]).$$

Hence, by appealing to well-known modulus estimates, we have

$$\begin{aligned} 0 < C(n) &\leq \operatorname{mod}(f\Delta_i, f[a_i, z_i]; \mathbf{B}^n) \\ &\leq K \operatorname{mod}(\Delta_i, [a_i, z_i]; D) \leq K \operatorname{mod}(\Delta_i, [a_i, z_i]; \mathbf{R}^n), \end{aligned}$$

so that

$$\operatorname{dist}(x_i, [a_i, z_i]) \leq C_5 d_i \tag{3.2}$$

for some $C_5 = C_5(n, K)$; see, for instance, [V1, 11.9; Vu, II.7].

Next we analyze possible locations of x_i. Suppose first that $x_i \in B(a_i', \lambda R_i)$, where $\lambda < (2C_4)^{-1}$ is a small positive constant, depending only on n and K, which will be adjusted repeatedly in the course of the proof. If ∂D meets $B(a_i', 2\lambda R_i)$, we have by (3.2) that

$$R_i - \lambda R_i \leq \operatorname{dist}(x_i, [a_i, z_i]) \leq C_5 d_i \leq C_5 4\lambda R_i,$$

which is impossible if $\lambda < (4C_5+1)^{-1}$. Thus we may assume that $B(a_i', 2\lambda R_i) \subset D$ if $x_i \in B(a_i', \lambda R_i)$. Then fix another small positive number $\varepsilon = \varepsilon(n, K) < 1$ whose value will be determined later. Join x_i to w_i by an arc α along $\partial B(a_i, R_i)\backslash B(b_i', R_i/C_4)$ such that $\operatorname{length}\alpha \leq 2\pi R_i$, and let x_i' be the first point on α for which

$$d_i' = \operatorname{dist}(x_i', \partial D) = \varepsilon\lambda R_i. \tag{3.3}$$

Because ∂D does not meet $B(a_i', 2\lambda R_i)$ and because $\varepsilon < 1$, the point x_i' lies outside $B(a_i', \lambda R_i)$. Further, it follows straight from the definition for the quasihyperbolic metric (see §5) that

$$k_D(x_i, x_i') \leq \frac{\operatorname{length}\alpha}{\varepsilon\lambda R_i} \leq \frac{2\pi}{\varepsilon\lambda} = C(n, K).$$

In particular, we deduce from the uniform continuity of quasiconformal maps in the quasihyperbolic metric [GO, Theorem 3] that

$$k_{\mathbf{B}^n}(f(x_i), f(x_i')) \leq C(n, K),$$

whence

$$1 - |f(x_i)| \leq C(n, K)(1 - |f(x_i')|). \tag{3.4}$$

Thus, if $x_i \in B(a_i', \lambda R_i)$, we replace it with a point x_i',

$$x_i' \in \partial B(a_i, R_i)\backslash B(a_i', \lambda R_i)\backslash B(b_i', \lambda R_i), \tag{3.5}$$

for which (3.3) and (3.4) hold.

As the next case, we consider the situation where

$$x_i \in \partial B(a_i, R_i)\backslash B(a_i', \lambda R_i)\backslash B(b_i', \lambda R_i).$$

If $d_i \leq \varepsilon\lambda R_i$, there will be no changes. If $d_i > \varepsilon\lambda R_i$, then as above we can find an arc joining x_i to a point x_i' along $\partial B(a_i, R_i)\backslash B(a_i', \lambda R_i)\backslash B(b_i', \lambda R_i)$ such that (3.3) and (3.4) hold. Then we replace x_i with x_i'.

We are left with the case $x_i \in B(b_i', \lambda R_i)$. This is divided into two subcases depending on whether

$$\partial D \cap B(b_i', 2\lambda R_i) = \varnothing \tag{3.6}$$

or

$$\partial D \cap B(b_i', 2\lambda R_i) \neq \varnothing. \tag{3.7}$$

If (3.6) occurs, then $d_i \geq \lambda R_i$, and using the "arc trick" once more, we replace x_i with a point x_i' such that (3.3)–(3.5) hold.

We pause here to divide the set $A = \{x_i : i = 1, 2, \dots\}$ into two disjoint subsets A_1 and A_2, where A_2 consists of those points x_i, which lie in $B(b_i', \lambda R_i)$ and for which (3.7) holds, and $A_1 = A\backslash A_2$. As regards to A_1, we can assume by the aforesaid that whenever $x_i \in A_1$, then

$$x_i \in \partial B(a_i, R_i)\backslash B(a_i', \lambda R_i)\backslash B(b_i', \lambda R_i) \tag{3.8}$$

and

$$d_i = \operatorname{dist}(x_i, \partial D) \leq \varepsilon\lambda R_i. \tag{3.9}$$

Only (3.8) and (3.9) are required in proving the desired finite overlapping of the balls $B_i = B(x_i, C_1 d_i)$ for $x_i \in A_1$. We do this next and deal with the residual case A_2 later.

3.10. *Case* A_1. Because $d_i \le \varepsilon\lambda R_i$, we have that

$$\begin{aligned}\operatorname{dist}(B_i, L) &\ge \operatorname{dist}(x_i, L) - C_1 d_i \ge \frac{\lambda R_i}{2} - C_1 d_i \\ &\ge \frac{\lambda R_i}{2} - C_1\varepsilon\lambda R_i = \lambda R_i\left(\frac{1}{2} - C_1\varepsilon\right) \ge \frac{\lambda R_i}{4},\end{aligned}$$

provided $\varepsilon < (4C_1)^{-1}$. On the other hand, it is easy to see that $B_i \subset B(a_i, 2R_i)$, so that B_i lies in the zone

$$Z_i = \{x \in \mathbf{R}^n : \frac{\lambda R_i}{4} \le \operatorname{dist}(x, L) \le 2R_i\}.$$

Suppose now that $x_i, x_j \in A_1$ with $i \ne j$. If the corresponding line segments L_i, L_j belong to the same generation $\mathscr{L}_\nu$, we have that

$$2^{-\nu-1} < R_j \le \operatorname{diam}\gamma_j \le C_2|a_j - z_j|$$

by Fact 2. Thus $|a_j - z_j| \ge C_2^{-1}2^{-\nu-1} \ge (2C_2)^{-1}R_i$, and we infer that there only can be $C(n, K)$ indices j such that $a_j \in B(a_i, 10R_i)$. On the other hand, if $a_j \notin B(a_i, 10R_i)$, then $B(a_j, 2R_j) \cap B(a_i, 2R_i) = \varnothing$ and hence $B_i \cap B_j = \varnothing$.

Next suppose that $L_i \in \mathscr{L}_{\nu_i}$ and $L_j \in \mathscr{L}_{\nu_j}$ with $\nu_i \ne \nu_j$. By symmetry we may assume that $\nu_j > \nu_i$. Then (3.1) gives

$$R_j = |a_j - x_j| \le 2^{-\nu_j} \le 2^{-N+2}R_i,$$

and by choosing N such that $2^{-N+3} < \lambda/4$, we have $2R_j < \lambda R_i/4$. Hence the zones Z_j and Z_i cannot meet. Consequently, the balls B_i and B_j cannot meet, and we can conclude the proof for the points in A_1.

3.11. *Case* A_2. Recall that the points in A_2 are those that belong to $B(b_i', \lambda R_i)$ with $B(b_i', 2\lambda R_i) \cap \partial D \ne \varnothing$. If $[a_i, z_i]$ does not meet $B(b_i', \sqrt{\lambda}R_i)$, then (3.2) implies

$$\sqrt{\lambda}R_i - \lambda R_i \le \operatorname{dist}(x_i, [a_i, z_i]) \le C_5 d_i \le C_5 4\lambda R_i,$$

which is impossible for $\lambda < (4C_5 + 1)^{-2}$. Hence we may assume that there is a point $z_i' \in [a_i, z_i] \cap B(b_i', \sqrt{\lambda}R_i)$. We have

$$\operatorname{dist}(a_i, B(b_i', 4\sqrt{\lambda}R_i)) \ge |a_i - b_i'| - 4\sqrt{\lambda}R_i = R_i(1 - 4\sqrt{\lambda}) \ge R_i/2$$

if $\lambda < 1/64$. Similarly, by Fact 2,

$$\begin{aligned}\operatorname{dist}(b_i, B(z_i', 3\sqrt{\lambda}R_i)) &\ge |b_i - z_i'| - 3\sqrt{\lambda}R_i \\ &\ge |b_i - z_i| - 3\sqrt{\lambda}R_i \ge |a_i - z_i| - 3\sqrt{\lambda}R_i \\ &\ge C_2^{-1}\operatorname{diam}\gamma_i - 3\sqrt{\lambda}R_i \ge R_i(C_2^{-1} - 3\sqrt{\lambda}) \ge (2C_2)^{-1}R_i\end{aligned}$$

if $\lambda < (6C_2)^{-2}$. A simple argument shows that

$$B_i \subset B(x_i, C_1 4\lambda R_i) \subset B(z_i', 3\sqrt{\lambda}R_i) \subset B(b_i', 4\sqrt{\lambda}R_i),$$

provided $\lambda < (4C_1)^{-2}$. We deduce that B_i is contained in a ball $B(z_i', 3\sqrt{\lambda}R_i)$ which is centered at a point on the line segment L_i and has positive distance to both a_i and b_i. Therefore, for points in A_2, the ball B_i cannot meet any B_j for $j \neq i$.

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

We only sketch the argument, which is similar to that in the proof of Theorem 1.2.

Suppose that (x_i) is a separated sequence on $L \cap D$. Some easy reasoning using the uniform continuity of quasiconformal maps in the quasihyperbolic metric shows that each point x_i may be assumed to be the center point of a line segment $L_i = [a_i, b_i] \subset L \cap D$ such that $|a_i - b_i| \leq M \operatorname{dist}(L_i, \partial D)$ for some $M = M(n, K)$. Let $S_i \subset \partial \mathbf{B}^n$ be the set, given by Fact 1, associated with $f(x_i)$. It is also proved in [HK, Lemma 6.6] that $S_i \subset B(f(x_i), C_1(1-|f(x_i)|))$, whence the Carleson measure condition (1.4) follows if we show that each point $x \in \mathbf{R}^n$ belongs to at most $C(n, K)$ balls $B_i = B(x_i, C_1 \operatorname{dist}(x_i, \partial D))$.

We may assume as in the proof of Theorem 1.2 that

$$L_i \in \mathscr{L}_{\nu_i} \text{ and } L_j \in \mathscr{L}_{\nu_j} \quad \text{implies} \quad \nu_i = \nu_j \text{ or } |\nu_i - \nu_j| \geq N \tag{4.1}$$

for some $N = N(n, K)$, where the generations are now defined by declaring that $L_i \in \mathscr{L}_\nu$ if and only if $2^{-\nu-1} < |a_i - b_i| \leq 2^{-\nu}$, $\nu \in \mathbf{Z}$. For each i, at least one of the balls $B(a_i, \frac{3}{4}|a_i - b_i|)$ or $B(b_i, \frac{3}{4}|a_i - b_i|)$ contains a point $w_i' \in \partial D$ such that $\operatorname{dist}(L_i, \partial D) = \operatorname{dist}(L_i, w_i')$, provided M is large enough; we can assume that $w_i' \in B(a_i, \frac{3}{4}|a_i - b_i|)$. Now replace x_i with the point where L_i meets $\partial B(a_i, \frac{3}{4}|a_i - b_i|)$, and call it still x_i. By using Fact 3, we can find a point

$$w_i \in \complement D \cap (\partial B(a_i, R_i) \backslash B(-x_i, \lambda R_i)),$$

where $\lambda = \lambda(n, K) > 0$ is a small constant, $R_i = |a_i - x_i|$, and $-x_i$ is the point on $\partial B(a_i, R_i)$ antipodal to x_i.

Next we divide (x_i) into two groups depending on whether ∂D meets $B(x_i, \lambda R_i)$ or not. If the first alternative occurs, then $B_i \subset B(x_i, C_1 \lambda R_i) \subset B(x_i, \frac{1}{4} R_i)$, provided $\lambda < (4C_1)^{-1}$, and hence no B_j can meet B_i for $i \neq j$. For points in the second group, we can use the arc trick as in the proof of Theorem 1.2 so as to find a point $x_i' \in \partial B(a_i, R_i) \backslash B(x_i, 2^{-1} \lambda R_i)$ such that $\operatorname{dist}(x_i', \partial D) = \varepsilon \lambda R_i$ for some $\varepsilon = \varepsilon(n, K) < \frac{1}{2}$ and that the quasihyperbolic distance between x_i and x_i' is less than $C(n, K)$. In this case we replace x_i with x_i' and show that only finitely many balls $B_i = B(x_i', C_1 \varepsilon \lambda R_i)$ can overlap. Indeed, if ε is sufficiently small, the balls B_i are contained in the zone $Z_i = \{x : 5^{-1} \lambda R_i \leq \operatorname{dist}(x, L) \leq 2R_i\}$, and therefore, if $B_i \cap B_j \neq \varnothing$, the corresponding intervals L_i and L_j must belong to the same generation $\mathscr{L}_\nu$, provided N is large enough. On the other hand, because $B_i \subset B(a_i, 2R_i)$, it is easy to see that at most $C(n, K)$ indices j are such that B_j meets B_i, should L_i and L_j belong to the same generation.

This completes the proof of Theorem 1.3.

5. Appendix: Quasihyperbolic geodesics up to the boundary

The *quasihyperbolic metric* in a proper subdomain D of $\mathbf{R}^n$ is defined by

$$k_D(x, y) = \inf \int_\gamma \frac{ds}{\operatorname{dist}(z, \partial D)},$$

where the infimum is taken over all rectifiable arcs γ in D joining x and y. There always exists a *quasihyperbolic geodesic* in D for which the infimum above is attained. See [GO] or [Vu] for the basic properties of this metric. Our proof of Theorem 1.2 required geodesics that ran to the boundary. The existence of such geodesics is probably folklore but nowhere in print, and for a possible future reference we consider here a somewhat more general situation that what was needed earlier in the paper.

We assume throughout that D is a proper subdomain of $\mathbf{R}^n$ with $n \geq 2$. By an *endcut* in D we mean an arc $\alpha \subset D$ such that $\bar{\alpha}$ is a compact arc with one end point on ∂D. The subarc of α between z and w is denoted by $\alpha[z, w]$.

We say that an endcut $\alpha \subset D$ is a *quasihyperbolic endcut* in D if every compact subarc of α is a quasihyperbolic geodesic in the usual sense. Now a boundary point need not be an end point of a quasihyperbolic endcut even if it is an end point of a straight endcut. As an example, consider the upper half plane in $\mathbf{R}^2$ with the line segments $[-1, 1] \times \{1/k\}$, $k = 1, 2, \ldots$, removed; then open little gates about the points $(0, 1/k)$ with width $\varepsilon_k > 0$. The entire positive x_2-axis lies in the resulting domain, but it is easy to see that no quasihyperbolic geodesic will travel through more than three gates, provided $\varepsilon_k \to 0$ fast enough. Thus the origin cannot be an end point of any geodesic endcut.

5.1. **Theorem.** *Suppose that D can be mapped onto a uniform domain via a quasiconformal homeomorphism, and suppose that $a \in \partial D$ is an end point of an endcut $\alpha \subset D$. Then each $b \in D$ can be joined to a by a quasihyperbolic endcut in D.*

Recall that D is *uniform* if there is a constant $c \geq 1$ such that each pair of points $z, w \in D$ can be joined by an arc $\beta \subset D$ satisfying

$$\operatorname{diam} \beta \leq c|z - w|$$

and

$$\min\{\operatorname{diam} \beta[z, x], \operatorname{diam} \beta[w, x]\} \leq c \operatorname{dist}(x, \partial D)$$

for all $x \in \beta$.

Proof of Theorem 5.1. Fix $b \in D$. We may assume that b and a are the end points of and endcut α. Let $a_i \in \alpha$ be such that $a_i \to a$, $i \to \infty$, and let γ_i be a quasihyperbolic geodesic joining b to a_i. Passing to a subsequence if necessary, we may assume that γ_i converges to a continuum $\gamma \subset \overline{D} \subset \mathbf{R}^n \cup \{\infty\}$ in the Hausdorff metric; see, e.g., [F, p. 37]. Clearly γ contains both a and b, and we claim that $\gamma \cap \partial D = \{a\}$. Assuming the contrary and relabeling if necessary, we can find a sequence of points $x_i \in \gamma_i$ such that $x_i \to z_0 \in \partial D \setminus \{a\}$. Suppose first that z_0 is not the point at infinity. Because D is quasiconformally equivalent to a uniform domain, [HN, Lemma 7.1 and Theorem 6.1] provide us with arcs β_i that join x_i to $\alpha[b, a_i]$ in D such that $\operatorname{diam} \beta_i \leq A \operatorname{dist}(x_i, \partial D)$, where A is independent of i. Thus

$$\begin{aligned} 0 < \operatorname{dist}(z_0, \alpha) &\leq |z_0 - x_i| + \operatorname{dist}(x_i, \alpha) \\ &\leq |z_0 - x_i| + \operatorname{diam} \beta_i \leq |z_0 - x_i| + A \operatorname{dist}(x_i, \partial D) \to 0, \end{aligned}$$

which is a contradiction. If $z_0 = \infty$, we use [HN, 7.1 and 6.1] to find arcs β_i as above, this time having the property that $\operatorname{diam} \beta_i \leq A \operatorname{diam} \alpha$. This again is impossible, and we conclude that $\gamma \cap \partial D = \{a\}$.

Next we show that γ is an arc. For this it suffices to show that each $z \in \gamma\backslash\{a, b\}$ is a cut point of γ; see [N, 4.10.2]. We borrow an argument from [GNV, Lemma 5.11]. Let (z_i) be a sequence in γ_i converging to $z \in \gamma\backslash\{a, b\}$. Then, after passing to a subsequence and relabeling if needed, we have that $\gamma_i[a_i, z_i]$ and $\gamma_i[z_i, b]$ converge to two continua Γ_1 and Γ_2, respectively. Since $\gamma_i[a_i, z_i] \cup \gamma_i[z_i, b] = \gamma_i$, the continuum γ is the union of Γ_1 and Γ_2. Suppose that there is a point $z' \in (\Gamma_1 \cap \Gamma_2)\backslash\{z\}$. Then z' is a limit of two sequences, say (x_i) in $\gamma_i[a_i, z_i]$ and (y_i) in $\gamma_i[z_i, b]$. This means that for sufficiently large indices i the points x_i and y_i all lie in an arbitrarily small ball about z' not containing the points z_i; because each z_i lies in between x_i and y_i on a quasihyperbolic geodesic γ_i, this is easily seen to be a contradiction. Therefore $\Gamma_1 \cap \Gamma_2 = \{z\}$, and it follows that $\gamma\backslash\{z\}$ is not connected. Thus z is a cut point of γ.

It remains to show that $\gamma[b, z]$ is a quasihyperbolic geodesic between b and z for $z \in \gamma\backslash\{a\}$. As above, choose a sequence (z_i) from γ_i converging to z. Because $k_D(b, z_i) \le M < \infty$ with M independent of i, the Euclidean lengths of the geodesics $\gamma_i[b, z_i]$ are uniformly bounded. By the well-known lower semicontinuity theorem [F, 3.18], we have that

$$\operatorname{length} \gamma[b, z] \le \liminf_{i\to\infty} \operatorname{length} \gamma_i[b, z_i],$$

which implies that the length of $\gamma[b, z]$ is finite. Finally, by the continuity of the density $\operatorname{dist}(x, \partial D)^{-1}$ (and by [F, 3.18]) it is easy to see that

$$\begin{aligned} k_D(b, z) &\le \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)} \le \lim_{i\to\infty} \int_{\gamma_i[b, z_i]} \frac{ds}{\operatorname{dist}(x, \partial D)} \\ &= \lim_{i\to\infty} k_D(b, z_i) = k_D(b, z). \end{aligned}$$

This completes the proof.

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