PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

BEST CONSTANTS FOR TWO NONCONVOLUTION INEQUALITIES

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(Communicated by J. Marshall Ash)

ABSTRACT. The norm of the operator which averages $|f|$ in $L^p(\mathbb{R}^n)$ over balls of radius $\delta|x|$ centered at either 0 or x is obtained as a function of n, p and δ. Both inequalities proved are n-dimensional analogues of a classical inequality of Hardy in $\mathbb{R}^1$. Finally, a lower bound for the operator norm of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^n)$ is given.

0. INTRODUCTION

A classical result of Hardy [HLP] states that if f is in $L^p(\mathbb{R}^1)$ for $p > 1$, then

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^\infty |f(t)|^p\,dt\right)^{1/p} \tag{0.1}$$

and the constant $p/(p-1)$ is the best possible. By considering two-sided averages of f instead of one-sided, (0.1) can be equivalently formulated as:

$$\left(\int_{-\infty}^\infty \left(\frac{1}{2|x|}\int_{-|x|}^{|x|} |f(t)|\,dt\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_{-\infty}^\infty |f(t)|^p\,dt\right)^{1/p}. \tag{0.2}$$

We denote by $D(a, R)$ the ball of radius R in $\mathbb{R}^n$ centered at a. Let $(Tf)(x)$ be the average of $|f| \in L^p(\mathbb{R}^n)$ over the ball $D(0, |x|)$. The analogue of (0.2) for $\mathbb{R}^n$ is the inequality:

$$\|Tf\|_{L^p} \leq C_p(n)\|f\|_{L^p} \tag{0.3}$$

for some constant $C_p(n)$ which depends a priori on p and n. Our first result is that the best constant $C_p(n)$ which satisfies (0.3) for all $f \in L^p(\mathbb{R}^n)$ is $p' = p/(p-1)$, the same constant as in dimension one. Another version of Hardy's inequality in $\mathbb{R}^n$ with the best possible constant can be found in [F].

Next we consider a similar problem. An equivalent formulation of (0.1) and (0.2) is

(0.4)
$$\left(\int_{-\infty}^\infty \left(\frac{1}{2|x|}\int_{x-|x|}^{x+|x|} |f(t)|\,dt\right)^p dx\right)^{1/p} \leq \frac{p}{2^{1/p}(p-1)}\left(\int_{-\infty}^\infty |f(t)|^p\,dt\right)^{1/p},$$

Received by the editors May 5, 1993 and, in revised form, September 3, 1993.
1991 *Mathematics Subject Classification*. Primary 42B25.
The authors' research was partially supported by the National Science Foundation.

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0002-9939/95 \$1.00 + \$.25 per page

where f is in $L^p(\mathbb{R}^1)$. Let $(Sf)(x)$ be the average of $|f| \in L^p(\mathbb{R}^n)$ over the ball $D(x, |x|)$. We compute the operator norm $c_{p,n}$ of S on $L^p(\mathbb{R}^n)$ as a function of n and p. The precise value of the constant $c_{p,n}$ is given in Theorem 2.

In section 3 a lower bound for the operator norm of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^n)$ is given. Finally, in section 4 the norm on $L^p(\mathbb{R}^n)$ of the operator which averages f over the ball of radius $\delta|x|$ centered at either 0 or $|x|$ is given as a function of δ, p, and n, for any $\delta > 0$.

Throughout this note, ω_{n-1} will denote the area of the unit sphere S^{n-1} and v_n the volume of the unit ball in $\mathbb{R}^n$.

1. Hardy's inequality on $\mathbb{R}^n$

In this section we will prove inequality (0.3) with constant $C_p(n) = p' = p/(p-1)$. We denote by $|A|$ the Lebesgue measure of the set A and by χ_A its characteristic function.

Theorem 1. *Let $f \in L^p(\mathbb{R}^n)$, where $1 < p < \infty$. The following inequality holds:*
(1.1)

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|D(0,|x|)|}\int_{D(0,|x|)}|f(y)|\,dy\right)^p dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_{\mathbb{R}^n}|f(y)|^p\,dy\right)^{1/p},$$

and the constant $p' = p/(p-1)$ is the best possible.

Proof. Fix $f \in L^p(\mathbb{R}^n)$. Without loss of generality, assume that f is nonnegative and continuous. Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers with Haar measure $\frac{dt}{t}$. The function $t^{n/p'}\chi_{[0,1]}$ is in $L^1(\mathbb{R}^+, \frac{dt}{t})$ with norm p'/n. For a fixed θ in the unit sphere S^{n-1}, the function $t \to f(t\theta)t^{n/p}$ is in $L^p(\mathbb{R}^+, \frac{dt}{t})$. The group inequality $\|g * K\|_{L^p} \le \|g\|_{L^p}\|K\|_{L^1}$ gives:

$$\int_{r=0}^{\infty}\left(\int_0^1 f(rt\theta)(rt)^{\frac{n}{p}}t^{\frac{n}{p'}}\frac{dt}{t}\right)^p\frac{dr}{r} \le \left(\int_{r=0}^{\infty}(f(r\theta)r^{\frac{n}{p}})^p\frac{dr}{r}\right)\left(\frac{p'}{n}\right)^p. \tag{1.2}$$

Note that equality holds in (1.2) if and only if equality holds in $\|g * K\|_{L^p} \le \|g\|_{L^p}\|K\|_{L^1}$. This happens in the limit by the sequence $g_{\epsilon,N} = \chi_{[\epsilon,N]}$. Since $g(t) = f(t\theta)t^{n/p}$, we conclude that equality is attained in (1.2) in the limit by the sequence

$$f_{\epsilon,N}(t\theta) = t^{-n/p}\chi_{\epsilon\le t\le N} \qquad \text{as } \epsilon \to 0 \text{ and } N \to \infty. \tag{1.3}$$

Note that Tf is a radial function. Expressing all integrals in polar coordinates, we reduce (1.1) to a convolution inequality on the multiplicative group $\mathbb{R}^+$. We have

$$\begin{aligned}\|Tf\|^p_{L^p(\mathbb{R}^n)} &= \omega_{n-1}\int_{r=0}^{\infty}\left(\frac{1}{v_n r^n}\int_{t=0}^{r}\int_{\theta\in S^{n-1}} f(t\theta)t^{n-1}\,d\theta dt\right)^p r^{n-1}\,dr\\ &= \frac{\omega_{n-1}}{v_n^p}\int_{r=0}^{\infty}\left(\int_{S^{n-1}}\int_{t=0}^{1} f(rt\theta)(rt)^{\frac{n}{p}}t^{\frac{n}{p'}}\frac{dt}{t}d\theta\right)^p\frac{dr}{r}.\end{aligned} \tag{1.4}$$

We apply Hölder's inequality with exponents $\frac{1}{p} + \frac{1}{p'} = 1$ to the functions 1 and $\theta \to \int_{t=0}^1 f(rt\theta)(rt)^{n/p}t^{n/p'}\frac{dt}{t}$ and then to Fubini's theorem to interchange the integrals in θ and r. We obtain that (1.4) is bounded above by

$$\frac{\omega_{n-1}}{v_n^p}\omega_{n-1}^{\frac{p}{p'}}\int_{S^{n-1}}\int_{r=0}^{\infty}\left(\int_{t=0}^{1} f(rt\theta)(rt)^{\frac{n}{p}}t^{\frac{n}{p'}}\frac{dt}{t}\right)^p\frac{dr}{r}\,d\theta. \tag{1.5}$$

Note that if f is a radial function, then (1.4) and (1.5) are identical. We now apply (1.2) to majorize (1.5) by

$$\frac{\omega_{n-1}^p}{v_n^p}\left(\frac{p'}{n}\right)^p \int_{S^{n-1}}\int_{r=0}^{\infty} f(r\theta)^p r^n \frac{dr}{r}\,d\theta = \left(\frac{p}{p-1}\right)^p \|f\|_{L^p(\mathbb{R}^n)}^p$$

using the fact that $\omega_{n-1} = nv_n$. We have now obtained the inequality $\|Tf\|_{L^p} \le p'\|f\|_{L^p}$. Equality holds when the family of functions (1.3) is radial. Therefore, the extremal family for inequality (1.1) is $|x|^{-n/p}\chi_{\epsilon\le|x|\le N}$, as $\epsilon \to 0$ and $N \to \infty$.

2. A variant of Hardy's inequality on $\mathbb{R}^n$

The derivation of the n-dimensional analogue of (0.4) is more subtle. Let $B(s, t)$ denote the usual beta-function $\int_0^1 x^t(1-x)^s\,dx$. Our second result is

Theorem 2. *Let $1 < p < \infty$ and $c_{p,n} = p'\frac{\omega_{n-2}}{\omega_{n-1}}2^{\frac{n}{p'}-1}B(\frac{1}{2}(\frac{n}{p'}-1), \frac{n-3}{2})$. The following inequality holds for all f in $L^p(\mathbb{R}^n)$:*

(2.1)

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|D(x, |x|)|}\int_{D(x,|x|)}|f(y)|\,dy\right)^p dx\right)^{1/p} \le c_{p,n}\left(\int_{\mathbb{R}^n}|f(y)|^p\,dy\right)^{1/p}$$

and the constant $c_{p,n}$ is the best possible.

Proof. We use duality. Fix f and g positive and continuous with $\|f\|_{L^p(\mathbb{R}^n)} \le 1$ and $\|g\|_{L^{p'}(\mathbb{R}^n)} \le 1$. We will show that $\int g(x)(Sf)(x)\,dx \le c_{p,n}$. We express both g and Sf in polar coordinates by writing $x = r\phi$ and $y = t\theta$. The relation $|x-y| \le |x|$ is equivalent to $\theta\cdot\phi \ge t/2r$. We obtain

(2.2)

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(x)(Sf)(x)\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{v_n|x|^n}f(y)g(x)\chi_{D(x,|x|)}(y)\,dx\,dy \\
&= \frac{1}{v_n}\iint_{(S^{n-1})^2}\int_{r=0}^{\infty}\int_{t=0}^{2r} f(t\theta)g(r\phi)\chi_{\phi\cdot\theta\ge t/2r}\, t^n\frac{dt}{t}\frac{dr}{r}\,d\phi\,d\theta \\
&= \frac{2^{\frac{n}{p'}}}{v_n}\iint_{(S^{n-1})^2}\int_{r=0}^{\infty} g(r\phi)r^{\frac{n}{p'}}\left(\int_{t=0}^{1} f(2rt\theta)(2rt)^{\frac{n}{p}}\chi_{\phi\cdot\theta\ge t}\, t^{\frac{n}{p'}}\frac{dt}{t}\right)\frac{dr}{r}\,d\phi\,d\theta \\
&\le \frac{2^{\frac{n}{p'}}}{v_n}\iint_{(S^{n-1})^2} G(\phi)\left[\int_{r=0}^{\infty}\left(\int_{t=0}^{1} f(2rt\theta)(2rt)^{\frac{n}{p}}\chi_{\phi\cdot\theta\ge t}\, t^{\frac{n}{p'}}\frac{dt}{t}\right)^p\frac{dr}{r}\right]^{1/p} d\phi\,d\theta,
\end{aligned}$$

where $G(\phi) = \left(\int_{r=0}^{\infty} g(r\phi)^{p'}r^n\frac{dr}{r}\right)^{1/p'}$. The bracketed expression in (2.2) is the L^p norm of the group $(\mathbb{R}^+, \frac{dt}{t})$ convolution of the function $t \to f(t\theta)t^{\frac{n}{p}}$ with the kernel $\chi_{[0,\theta\cdot\phi]}(t)t^{\frac{n}{p'}}$ at $2r$. We therefore estimate (2.2) by

$$\frac{2^{\frac{n}{p'}}}{v_n}\iint_{(S^{n-1})^2} G(\phi)F(\theta)\left(\int_0^{\theta\cdot\phi} t^{\frac{n}{p'}}\frac{dt}{t}\right)d\phi\,d\theta, \tag{2.3}$$

where $F(\theta) = \left(\int_0^{\infty} f(r\theta)^p r^n\frac{dr}{r}\right)^{1/p}$. Let

$$K(\phi\cdot\theta) = \int_0^{\theta\cdot\phi} t^{n/p'}\frac{dt}{t} = \frac{p'}{n}[(\phi\cdot\theta)^+]^{n/p'},$$

where N^+ denotes the positive part of the number N. Next, we need the following:

Lemma. *For any $F, G \geq 0$ measurable on S^{n-1} and $K \geq 0$ measurable on* $[-1, 1]$,

$$\tag{2.4} \begin{aligned} &\iint_{(S^{n-1})^2} F(\theta)\, G(\phi)\, K(\theta \cdot \phi)\, d\phi\, d\theta \\ &\qquad \leq \|F\|_{L^p(S^{n-1})} \|G\|_{L^{p'}(S^{n-1})} \int_{S^{n-1}} K(\theta \cdot \phi)\, d\phi. \end{aligned}$$

Proof. We may assume that all three quantities on the right-hand side of (2.4) are finite. Since K depends only on the inner product $\theta \cdot \phi$, the integral $\int_{S^{n-1}} K(\theta \cdot \phi)\, d\phi$ is independent of θ. Hölder's inequality applied to the functions F and 1 with respect to the measure $K(\theta \cdot \phi)\, d\theta$ gives

$$\tag{2.5} \begin{aligned} &\int_{S^{n-1}} F(\theta) K(\theta \cdot \phi)\, d\theta \\ &\qquad \leq \left(\int_{S^{n-1}} F(\theta)^p K(\theta \cdot \phi)\, d\theta \right)^{1/p} \left(\int_{S^{n-1}} K(\theta \cdot \phi)\, d\theta \right)^{1/p'}. \end{aligned}$$

We will now use (2.5) to prove (2.4). The left-hand side of (2.4) is trivially estimated by $\left(\int_{S^{n-1}} \left(\int_{S^{n-1}} F(\theta) K(\theta \cdot \phi)\, d\theta \right)^p d\phi \right)^{1/p} \|G\|_{L^{p'}(S^{n-1})}$. Applying (2.5) and Fubini's theorem we bound this last expression by $\|F\|_{L^p(S^{n-1})} \|G\|_{L^{p'}(S^{n-1})} \times \int_{S^{n-1}} K(\theta \cdot \phi)\, d\phi$. The lemma is now proved. Observe that equality is attained in (2.4) if and only if both F and G are constants.

We now continue with the proof of Theorem 2. Applying the lemma and using the fact that F and G have norm one, we estimate (2.3) by $\frac{p'}{n} \frac{2^{\frac{n}{p'}}}{v_n} \times \int_{S^{n-1}} \left((\theta \cdot \phi)^+ \right)^{\frac{n}{p'}} d\theta$. To compute this integral, we slice the sphere in the direction transverse to ϕ. For convenience we may take $\phi = e_1 = (1, 0, \dots, 0)$. The area of the slice cut by the hyperplane $\phi_1 = s$ is $\omega_{n-2}(1 - s^2)^{\frac{n-2}{2}}$ and the weight of this slice is $(1 - s^2)^{-\frac{1}{2}}$. We get

$$\tag{2.6} \begin{aligned} \int_{S^{n-1}} \left((\theta \cdot \phi)^+ \right)^{\frac{n}{p'}} d\theta &= \omega_{n-2} \int_{s=0}^{1} s^{\frac{n}{p'}} (1 - s^2)^{\frac{n-3}{2}}\, ds \\ &= \omega_{n-2}\, \tfrac{1}{2}\, B\left(\tfrac{1}{2}\left(\tfrac{n}{p'} - 1\right), \tfrac{n-3}{2} \right). \end{aligned}$$

We now use that $n v_n = \omega_{n-1}$ to get the final estimate $c_{p,n}$ in (2.2) which completes the proof of (2.1). It remains to establish that the constant $c_{p,n}$ is the best possible. For any $y \in \mathbb{R}^n$, let $A(y)$ be the spherical cap $\{\theta \in S^{n-1} : |\theta - y| \leq |y|\}$. This cap is nonempty if and only if $|y| \geq 1/2$. For such y, the Lebesgue measure $|A(y)|$ is $\omega_{n-2} \int_{1/2|y|}^{1} (1 - s^2)^{\frac{n-3}{2}}\, ds$. Let $G(t) = \chi_{[0,1]}(t)\, t^{n/p'} \int_t^1 (1 - s^2)^{\frac{n-3}{2}}\, ds$. An easy computation shows that $\|G\|_{L^1(\mathbf{R}^+, \frac{dt}{t})} = \left(\frac{p'}{n}\right) \int_0^1 (1 - s^2)^{\frac{n-3}{2}} s^{\frac{n}{p'}}\, ds$. Let $h = h_{\epsilon, N}$ be an element of the family

$|x|^{-n/p}\chi_{\epsilon\le|x|\le N}$ normalized to have L^p norm one. We have

$$\begin{aligned}\|Sh\|^p_{L^p(\mathbb{R}^n)} &= \int_{r=0}^{\infty}\int_{\phi\in S^{n-1}}\left(\frac{1}{v_n r^n}\int_{D(r\phi,r)} h(y)\,dy\right)^p r^{n-1}\,d\phi\,dr\\ &= \int_{r=0}^{\infty}\int_{\phi\in S^{n-1}}\left(\frac{1}{v_n r^n}\int_{t=0}^{2r}\int_{\substack{\theta\in S^{n-1}\\ t\theta\in D(r\phi,r)}} h(t\theta)t^{n-1}\,d\theta dt\right)^p r^{n-1}\,d\phi\,dr\\ &= \int_{r=0}^{\infty}\int_{\phi\in S^{n-1}}\left(\frac{1}{v_n r^n}\int_{t=0}^{2r}|A((r/t)\phi)|h(t)t^n\,\frac{dt}{t}\right)^p r^n\,d\phi\,\frac{dr}{r}\end{aligned}$$

$$= \omega_{n-2}^p\frac{2^{np-n}}{v_n^p}\omega_{n-1}\int_{r=0}^{\infty}\left(\int_{t=0}^{1} h(2rt)(2rt)^{\frac{n}{p}}G(t)\,\frac{dt}{t}\right)^p r^n\,\frac{dr}{r}. \tag{2.7}$$

The convolution inequality $\|g*L\|_{L^p}\le\|g\|_{L^p}\|L\|_{L^1}$ in the group $(\mathbb{R}^+,\frac{dt}{t})$ written as

$$\int_{r=0}^{\infty}\left(\int_{t=0}^{1} h(2rt)(2rt)^{\frac{n}{p}}G(t)\,\frac{dt}{t}\right)^p\frac{dr}{r}\le\left(\int_{r=0}^{\infty}h(r)^p r^n\,\frac{dr}{r}\right)\|G\|^p_{L^1(\mathbb{R}^+,\frac{dt}{t})} \tag{2.8}$$

becomes an equality as $\epsilon\to 0$ and $N\to\infty$. Inserting (2.8) in (2.7) we obtain

$$\begin{aligned}\|Sh\|^p_{L^p(\mathbb{R}^n)} &\le \omega_{n-2}^p\frac{2^{np-n}}{v_n^p}\left(\frac{p'}{n}\right)^p\left(\int_{s=0}^{1}(1-s^2)^{\frac{n-3}{2}}s^{\frac{n}{p'}}\,ds\right)^p\\ &\quad\times\omega_{n-1}\int_{r=0}^{\infty}h(r)^p r^{n-1}dr = c^p_{p,n}\end{aligned}$$

since $\|h\|_{L^p}=1$, and equality is attained as $\epsilon\to 0$ and $N\to\infty$. Theorem 2 is now proved.

3. A lower bound for the operator norm of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^n)$

Let $M(f)(x) = \sup_{r>0}(v_n r^n)^{-1}\int_{|y-x|\le r}|f(y)|\,dy$ be the usual Hardy-Littlewood maximal function on $\mathbb{R}^n$. The family of functions $f_{\epsilon,N}(x) = |x|^{-n/p}\chi_{\epsilon\le|x|\le N}$ is extremal for Theorems 1 and 2. Let $A_{p,n}$ be the operator norm of M on $L^p(\mathbb{R}^n)$. By computing $\|M(f_{\epsilon,N})\|_{L^p}/\|f_{\epsilon,N}\|_{L^p}$ and letting $\epsilon\to 0$ and $N\to\infty$ we obtain a lower bound for $A_{p,n}$.

Proposition. *For $1<p<\infty$, let $A_{p,n}$ be the best constant C that satisfies the inequality $\|Mf\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}$ for all f in L^p. Then*

$$A_{p,n}\ge p'\,\frac{\omega_{n-2}}{\omega_{n-1}}\sup_{\delta>1}\frac{1}{\delta^n}\int_{-1}^{1}\left(\sqrt{1-s^2}\right)^{n-3}\left(s+\sqrt{s^2+\delta^2-1}\right)^{\frac{n}{p'}}ds \tag{3.1}$$

and the supremum above is attained for some $\delta=\delta_{n,p}$ always less than 2.

Proof. The following is only a sketch. Since $|x|^{-n/p}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we can calculate $M(|x|^{-n/p})$ instead. Observe that $M(|x|^{-n/p}) = c\,|x|^{-n/p}$ where $c = M(|x|^{-n/p})(e_1)$ and $e_1=(1,0,\dots,0)$. Also note that the supremum of the averages of $|x|^{-n/p}$ over balls of radius r centered at e_1 is attained for some $r=1+\gamma$ where $\gamma>0$. We therefore find that

$$c=\sup_{\gamma>0}\frac{1}{v_n(1+\gamma)^n}\int_{r=0}^{2+\gamma}r^{n-\frac{n}{p}}A_r\,\frac{dr}{r}, \tag{3.2}$$

where $A_r = |\{\theta \in S^{n-1} : |r\theta - e_1| < 1+\gamma\}|$. Calculation gives that $A_r = \omega_{n-1}$ for $r \le \gamma$ and $A_r = \omega_{n-2}\int_{(r^2-\gamma^2-2\gamma)/2r}^{1}(1-s^2)^{\frac{n-3}{2}}\,ds$ for $2+\gamma > r > \gamma$. We plug these values into (3.2), and we interchange the integration in r and s:

$$\int_{r=\gamma}^{2+\gamma}\int_{s=\frac{r^2-\gamma^2-2\gamma}{2r}}^{1} r^{\frac{n}{p'}}(1-s^2)^{\frac{n-3}{2}}\,ds\frac{dr}{r} = \int_{-1}^{1}\int_{r=\gamma}^{s+\sqrt{s^2+\gamma^2+2\gamma}} r^{\frac{n}{p'}}(1-s^2)^{\frac{n-3}{2}}\,\frac{dr}{r}ds.$$

We now let $\delta = \gamma+1$ and obtain (3.1). Note that the constant on the right-hand side of (3.1) reduces to the constant $c_{p,n}$ of Theorem 2 when $\delta = 1$.

4. Final remarks

We end with a couple of remarks. Let $c_{n,p}$ be the constant of Theorem 2. We observe that $c_{n,p} \le \frac{p}{p-1}$. This can be shown directly or via the following inequality which can be found in [HLP]:

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx \le \int_{\mathbb{R}^n} \tilde{f}(x)\tilde{g}(x)\,dx, \tag{4.1}$$

where f and g are integrable and $\tilde{f}$ denotes the symmetric decreasing rearrangement of any function f. Let T and S be the operators of Theorems 1 and 2. The nonsymmetric decreasing rearrangement of the kernel of S is the kernel of T. Taking g to be the kernel of S and f in $L^p \cap L^1$ in (4.1), we obtain the pointwise inequality $Sf \le T\tilde{f}$. It follows that $c_{n,p} \le \frac{p}{p-1}$.

For any $\delta > 0$, we define variants T_δ of T and S_δ of S as follows:

$$(T_\delta f)(x) = \frac{1}{|D(0,\delta|x|)|}\int_{D(0,\delta|x|)} f(y)\,dy$$

and

$$(S_\delta f)(x) = \frac{1}{|D(x,\delta|x|)|}\int_{D(x,\delta|x|)} f(y)\,dy.$$

Since $(T_\delta f)(x) = (Tf)(\delta x)$, it is immediate that the operator norm of T_δ on $L^p(\mathbb{R}^n)$ is $\frac{p}{p-1}\delta^{-n/p}$.

To compute the operator norm of S_δ on $L^p(\mathbb{R}^n)$, we repeat the proof of Theorem 2 verbatim. We obtain the following result:

Theorem. (A) *For $\delta > 1$, the operator norm of S_δ on $L^p(\mathbb{R}^n)$ is*

$$p'\,\frac{\omega_{n-2}}{\omega_{n-1}}\,\frac{1}{\delta^n}\int_{-1}^{1}(1-s^2)^{\frac{n-3}{2}}\left(s+\sqrt{s^2+\delta^2-1}\right)^{\frac{n}{p'}}ds.$$

(B) *For $\delta < 1$, the operator norm of S_δ on $L^p(\mathbb{R}^n)$ is*

$$p'\,\frac{\omega_{n-2}}{\omega_{n-1}}\,\frac{1}{\delta^n}\int_{s=\sqrt{1-\delta^2}}^{1}(1-s^2)^{\frac{n-3}{2}}\left[\left(s+\sqrt{s^2+\delta^2-1}\right)^{\frac{n}{p'}}-\left(s-\sqrt{s^2+\delta^2-1}\right)^{\frac{n}{p'}}\right]ds.$$

(3.1) is of course subsumed in conclusion (A) above.

Acknowledgment

The second author would like to thank Professor Al Baernstein for stimulating his interest in these problems and also for many useful conversations.

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