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A NOTE ON MORITA EQUIVALENCE OF TWISTED C^*-DYNAMICAL SYSTEMS

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ABSTRACT. We present an elementary proof that every twisted C^*-dynamical system is Morita equivalent to an ordinary system. As a corollary we prove the equivalence $C_0(G/H, A) \times_{\tilde{\alpha}, \tilde{u}} G \sim A \times_{\alpha, u} H$ for Busby-Smith twisted dynamical systems, generalizing an important result of Green.

It is essentially the content of a recent theorem of Echterhoff [5, Theorem 1] that every twisted dynamical system is Morita equivalent to an ordinary system. By avoiding Green's imprimitivity theorem [6, Corollary 5] and appealing directly to the stabilization trick [8, Theorem 3.4] of Packer and Raeburn, we provide an elementary proof of this fact, at the same time generalizing it (in the separable case) to Busby-Smith twisted systems. Thus our main theorem provides a way of lifting much of the theory for ordinary and Green-twisted systems to the more general systems. As an example of its utility, we prove an analog of Green's important equivalence $C^*(G, C_0(G/H, A); \tilde{\tau}) \sim C^*(H, A, \tau)$ [6] for Busby-Smith twisted systems. This, in turn, will form the basis for a process of inducing representations, an imprimitivity theorem, and ultimately a version of the Mackey-Green machine for these twisted systems.

1. Preliminaries

Throughout this note G will be a second-countable locally compact group; A and B will always be separable C^*-algebras. The multiplier algebra of A is denoted by $\mathcal{M}(A)$ and its unitary group by $\mathcal{UM}(A)$. If C^*-algebras A and B are (strongly) Morita equivalent via an equivalence bimodule X (see [10, 11]), we will write $A \sim_X B$ or simply $A \sim B$.

A *twisted action* of a group G on a C^*-algebra A is a pair (α, u) consisting of a strongly Borel map $\alpha: G \to \mathrm{Aut}(A)$ and a strictly Borel map $u: G \times G \to \mathcal{UM}(A)$ such that $\alpha_s \circ \alpha_t = \mathrm{Ad}\, u(s, t) \circ \alpha_{st}$ and $\alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$ for all r, s, t in G. We call the quadruple (A, G, α, u) a (*Busby-Smith*) *twisted dynamical system.* (See [2; 8, Definition 2.1].) If the cocycle u is trivial (i.e., identically 1), then we say (A, G, α, u) is an *ordinary*

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dynamical system and write (A, G, α) for short. Note that then α is a Borel homomorphism into the Polish group $\mathrm{Aut}(A)$, so is actually continuous [7, Proposition 5]. Thus the ordinary systems are the objects studied in [3] and [4], among others. For every twisted dynamical system there is a (unique) crossed product C^*-algebra $A \times_{\alpha, u} G$ which is a universal object for covariant representations of (A, G, α, u) in multiplier algebras [9].

Two twisted actions (β, v) and (α, u) of G on A are *exterior equivalent* if there is a strictly Borel map $w: G \to \mathcal{UM}(A)$ such that $\alpha_s = \mathrm{Ad}\, w_s \circ \beta_s$ and $u(s, t) = w_s\beta_s(w_t)v(s, t)w_{st}^*$ for any s, t in G [8, Definition 3.1].

More generally, suppose (B, G, β, v) and (A, G, α, u) are twisted systems and X is a $B - A$ equivalence bimodule. Let $\mathrm{Aut}(X)$ denote the set of bicontinuous linear bijections ϕ of X which satisfy the ternary homomorphism identity $\phi(x \cdot \langle y, z\rangle_A) = \phi(x) \cdot \langle \phi(y), \phi(z)\rangle_A$. (The analogous identity using B-valued inner products is equivalent.) Then following [1, Definition 2.1], we will say (B, G, β, v) and (A, G, α, u) are *Morita equivalent* if there is a strongly Borel map $\gamma: G \to \mathrm{Aut}(X)$ such that for s, t in G and x, y in X:

(1) $\alpha_s(\langle x, y\rangle_A) = \langle \gamma_s(x), \gamma_s(y)\rangle_A$.
(2) $\beta_s({}_B\langle x, y\rangle) = {}_B\langle \gamma_s(x), (x, y)\rangle$.
(3) $\gamma_s \circ \gamma_t(x) = v(s, t) \cdot \gamma_{st}(x) \cdot u(s, t)^*$.

We write $(B, G, \beta, v) \sim_{X, \gamma} (A, G, \alpha, u)$ and call (X, γ) a *system of imprimitivity* implementing the equivalence.

2. The main theorem

Theorem 2.1. *Let (A, G, α, u) be a twisted dynamical system, and let $\mathcal{K}$ denote the compact operators on $\mathcal{H} = L^2(G)$. Then there is an ordinary action β of G on $A \otimes \mathcal{K}$ and a map $\delta: G \to \mathrm{Aut}(A \otimes \mathcal{H})$ such that*

$$(A \otimes \mathcal{K}, G, \beta) \sim_{A\otimes\mathcal{H}, \delta} (A, G, \alpha, u).$$

Proof. We appeal to the Packer-Raeburn stabilization trick [8, Theorem 3.4]. Thus we have a Borel map $w: G \to \mathcal{UM}(A \otimes \mathcal{K})$ which implements an exterior equivalence between an ordinary action $(\beta, 1)$ of G on $A \otimes \mathcal{K}$ and $(\alpha \otimes \mathrm{id}_{\mathcal{K}}, u \otimes 1)$.

Let $A \otimes \mathcal{H}$ have the canonical $A \otimes \mathcal{K} - A$ equivalence bimodule structure; so $A \otimes \mathcal{H}$ is the completion of the algebraic tensor product $A \odot \mathcal{H}$ with respect to the norm induced by the A-valued inner product $\langle a \otimes \xi, b \otimes \eta\rangle_A = \langle \eta, \xi\rangle_{\mathcal{H}} a^* b$. For s in G, the rule $a \otimes \xi \mapsto \alpha_s(a) \otimes \xi$ defines an automorphism of $A \odot \mathcal{H}$ which satisfies condition (1) for this product, so is isometric with respect to the induced norm, and thus extends to a map $\alpha_s \otimes \mathrm{id}_{\mathcal{H}}$ of $A \otimes \mathcal{H}$ into itself. Then for x in $A \otimes \mathcal{H}$, the map $s \mapsto \alpha_s \otimes \mathrm{id}_{\mathcal{H}}(x)$ is Borel, using the fact that $s \mapsto \alpha_s(a)$ is Borel for a in A, together with a routine density argument.

Now define $\delta_s: A \otimes \mathcal{H} \to A \otimes \mathcal{H}$ by

$$\delta_s(x) = w_s^* \cdot \alpha_s \otimes \mathrm{id}_{\mathcal{H}}(x).$$

Then straightforward calculations on elementary tensors in $A \otimes \mathcal{H}$ verify that each δ_s satisfies the ternary homomorphism identity, and that the map $s \mapsto \delta_s$ satisfies conditions (1)–(3) above. For example, for any s, t in G and $a \otimes \xi$

in $A\otimes\mathscr{K}$ we have

$$\begin{aligned}\delta_s\circ\delta_t(a\otimes\xi) &= w_s^*\cdot\alpha_s\otimes\mathrm{id}_{\mathscr{K}}(w_t^*\cdot\alpha_t\otimes\mathrm{id}_{\mathscr{K}}(a\otimes\xi))\\ &= w_s^*\alpha_s\otimes\mathrm{id}_{\mathscr{K}}(w_t)^*\cdot\alpha_s\circ\alpha_t(a)\otimes\xi\\ &= (w_s^*\alpha_s\otimes\mathrm{id}_{\mathscr{K}}(w_t)^*u(s,t)\otimes 1)\cdot\alpha_{st}\otimes\mathrm{id}_{\mathscr{K}}(a\otimes\xi)\cdot u(s,t)^*\\ &= w_{st}^*\cdot\alpha_{st}\otimes\mathrm{id}_{\mathscr{K}}(a\otimes\xi)\cdot u(s,t)^*\\ &= \delta_{st}(a\otimes\xi)\cdot u(s,t)^*.\end{aligned}$$

In particular, condition (1) (or (2)) implies each δ_s is isometric, and therefore bicontinuous since each δ_s is invertible. Thus each δ_s belongs to $\mathrm{Aut}(A\otimes\mathscr{K})$.

It only remains to show that the map $s\mapsto\delta_s$ is strongly Borel. To see this, fix x in $A\otimes\mathscr{K}$ and let $\{e_i\}$ be a countable approximate identity for $A\otimes\mathscr{K}$. Then each of the maps $s\mapsto w_s^*e_i$ and $s\mapsto\alpha_s\otimes\mathrm{id}_{\mathscr{K}}(x)$ are Borel, so that $s\mapsto w_s^*e_i\cdot a_s\otimes\mathrm{id}_{\mathscr{K}}(x)$ is Borel for each i. Since $s\mapsto\delta_s(x)$ is the pointwise limit of these Borel maps, it too is Borel, and the theorem follows. □

We remark that the analogous theorem for separable Green-twisted systems can be derived from Theorem 2.1, proving in essence Echterhoff's [5, Theorem 1]. This just requires verifying that the process of converting Green-twisted systems into Busby-Smith systems described in [8, §5] preserves the respective notions of Morita equivalence. While not complicated, the proof is lengthy, so we will not include it here.

Now let H be a closed subgroup of G, and denote an element tH of the quotient space G/H by $\dot{t}$. We define the diagonal twisted action $(\tilde{\alpha},\tilde{u})$ of G on $C_0(G/H,A)$ as follows:

$$\tilde{\alpha}_s(f)(\dot{t})=\alpha_s(f(s^{-1}\dot{t}))\quad\text{and}\quad[\tilde{u}(s,t)f](\dot{r})=u(s,t)f(\dot{r}).$$

Then we have the promised analog of Green's result [6, Corollary 5].

Corollary 2.2. *Let (A,G,α,u) be a twisted dynamical system, and let H and $(\tilde{\alpha},\tilde{u})$ be as above. Then*

$$C_0(G/H,A)\times_{\tilde{\alpha},\tilde{u}}G\sim A\times_{\alpha,u}H.$$

Proof. Let $(A\otimes\mathscr{K},G,\beta)$ be the ordinary system which by Theorem 2.1 is Morita equivalent to (A,G,α,u). Then it is straightforward to check that $C_0(G/H,A\otimes\mathscr{K})$ is a $C_0(G/H,A\otimes\mathscr{K})-C_0(G/H,A)$ equivalence bimodule when equipped with pointwise actions and inner products. Moreover, calculations similar to those in the proof of Theorem 2.1 verify that the diagonal action $\tilde{\delta}$ of G on $C_0(G/H,A\otimes\mathscr{K})$ defined by $\tilde{\delta}_s(x)(\dot{t})=\delta_s(x(s^{-1}\dot{t}))$ yields

$$(C_0(G/H,A\otimes\mathscr{K}),G,\tilde{\beta})\sim_{C_0(G/H,A\otimes\mathscr{K}),\tilde{\delta}}(C_0(G/H,A),G,\tilde{\alpha},\tilde{u}).$$

Because Morita equivalent twisted systems have Morita equivalent crossed products [1, Theorem 2.3], we have $C_0(G/H,A)\times_{\tilde{\alpha},\tilde{u}}G\sim C_0(G/H,A\otimes\mathscr{K})\times_{\tilde{\beta}}G$.

Next, notice that restricting the twisted actions (α,u) and $(\beta,1)$ to H yields Morita equivalent systems (A,H,α,u) and $(A\otimes\mathscr{K},H,\beta)$. Again using [1, Theorem 2.3], we have $A\otimes\mathscr{K}\times_\beta H\sim A\times_{\alpha,u}H$. But Green's [6, Corollary 5] applied to $(A\otimes\mathscr{K},G,\beta)$ gives us $C_0(G/H,A\otimes\mathscr{K})\times_{\tilde{\beta}}G\sim A\otimes\mathscr{K}\times_\beta H$; the corollary now follows by the transitivity of Morita equivalence. □

The development of a theory of induced representations for Busby-Smith twisted systems—in particular, for the Mackey-Green machine—will require not only the abstract Morita equivalence of Corollary 2.2 but a concrete equivalence bimodule which implements it. To be sure, the use of transitivity in the above proof implicitly gives such a bimodule; namely, the balanced tensor product of the three bimodules involved. This three-fold tensor product bimodule turns out to be extremely unpleasant to work with. We would prefer a bimodule completion of $B_c(G, A)$, the bounded Borel functions with compact support, which would be analogous to Green's $C_c(G, A)$. Such a bimodule does exist; however, technical difficulties arise in proving this which are beyond the scope of this note. The general process of inducing covariant representations of Busby-Smith twisted dynamical systems, as well as the particular problem of providing a workable equivalence bimodule for Corollary 2.2 are addressed in work currently in preparation.

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