DYNAMICS OF TYPICAL CONTINUOUS FUNCTIONS

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ABSTRACT. S. J. Agronsky, A. M. Bruckner, and M. Laczkovic have studied the behaviour of the sequence $(f^n(x))$ where f is the typical continuous function from the closed unit interval I into itself and x the typical point of I . In particular, they have proved that the typical limit set $\omega(f, x)$ is a Cantor set of Menger-Uryson dimension zero. Using mainly the Tietze extension theorem, we have found a shorter proof of this result which applies to a more general situation. As a matter of fact, we have replaced the closed unit interval by a compact N -dimensional manifold and the Menger-Uryson dimension by the Hausdorff one. We have also proved that, for the typical continuous function f , the function $x \rightarrow \omega(f, x)$ is continuous at the typical point x . It follows that the typical limit set is not a fractal and that, for the typical continuous function f , the sequence $(f^n(x))$ is not chaotic.

INTRODUCTION

From now onward, we set X to denote a compact manifold with boundary, N to denote its dimension, and d to denote a metric on X such that X is a second countable complete metric space (i.e., having a countable basis). If B is a subset of X , we shall represent its interior by $\text{Int}(B)$, its closure by $\text{Clos}(B)$, its exterior (i.e., $X - \text{Clos}(B)$) by $\text{Ext}(B)$, its boundary by ∂B , and its diameter by $d(B)$.

$K(X)$ will denote the set of compact subsets of X and d_K the Hausdorff metric on $K(X)$ (see [5, p. 96]). Thus, $K(X)$ is a metric space.

$C(X)$ will denote the space of continuous functions from X into itself and d_∞ the metric of uniform convergence on $C(X)$. Thus, $C(X)$ is a complete metric space. With the metric $\delta = \max(d_\infty, d)$, $C(X) \times X$ is also a complete metric space. If $(f, x) \in C(X) \times X$, we shall denote by $\omega(f, x)$ the limit set (i.e., the set of the limit points) of the sequence $(f^n(x))$. Thus ω is a function from $C(X) \times X$ into $K(X)$.

If E is a complete metric space, we shall call residual a subset of E which contains a countable intersection of dense, open subsets. If $P(x)$ is a property, the sentences "for x typical in E , $P(x)$ ", "if x is typical in E , $P(x)$ ", and " $P(x)$ at the typical point x of E " will mean "there is a residual subset A of

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E such that, if x belongs to A , then $P(x)$ ". In the same situation, we may also say that " $P(x)$ is typical in E " or " $P(x)$ is typical".

LEMMAS

The proofs of our theorems rely on two lemmas which mainly use the Tietze extension theorem (in a particular version, see [2, Corollary 1, p. 82]).

Lemma 1. *Let $\alpha > 0$, $\beta > 0$, and let $E_{\alpha, \beta}$ be the set of those $(f, x) \in C(X) \times X$ such that there is a finite number of subsets $(k_i)_i$ verifying:*

- $d(k_i) < \beta$ for each i ;
- $\omega(f, x) \subset \bigcup_i k_i$ and $\sum_i d(k_i)^\alpha < 1$.

The set $E_{\alpha, \beta}$ contains an open, dense set in $C(X) \times X$.

Proof. Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. Using the compactness of X , we construct a finite cover of open sets of diameters strictly less than ε having a closure homeomorphic to I^N where I is the closed unit interval.

Let p be the smallest positive integer such that there are an integer $n < p$ and an element V of the cover containing $f^p(a)$ and $f^n(a)$. Let $m = p - n$. This property implies that the points $a, f(a), f^2(a), \dots, f^{n+m-1}(a)$ are distinct.

Let A be a closed subset, B and C two open subsets, and $y \notin A$ such that $f(y) \in B$ and $y \in C$. Using the continuity of f at the point y , we can construct a compact subset K homeomorphic to I^N such that: $K \cap A = \emptyset$, $K \subset C$, $y \in \text{Int}(K)$, $d(K) < \varepsilon$, and $f(K) \subset B$.

We use this property for $A = \{a, f(a), \dots, f^{n+m-2}(a)\}$, $y = f^{n+m-1}(a)$, $B = V$, and $C = X$, and obtain a compact K_{n+m-1} . Using it for $A = \{a, f(a), \dots, f^{n+m-3}(a)\} \cup K_{n+m-1}$, $y = f^{n+m-2}(a)$, $B = \text{Int}(K_{n+m-2})$, and $C = X$, we obtain K_{n+m-2} . We continue in the same way (with the exception of $C = V$ for K_n). Eventually, we have constructed a sequence of compacts $K_0, K_1, \dots, K_{n+m-1}$ homeomorphic to I^N such that:

- $a \in \text{Int}(K_0)$;
- $d(K_i) < \varepsilon$ and $f(K_i) \subset \text{Int}(K_{i+1})$ for $0 \leq i \leq n + m - 2$;
- $K_i \cap K_j = \emptyset$ for $i \neq j$;
- $K_n \subset V$ and $f(K_{n+m-1}) \subset V$.

Let $\rho > 0$ be small enough that $m\rho^\alpha < 1$ and $\rho < \beta$. For each i , we choose a compact k_i with a non-empty interior included in $\text{Int}(K_i)$ such that $d(k_i) < \rho$ and a point b_i in $\text{Int}(k_i)$.

For each i , let $\eta_i > 0$ be such that the ball centred at b_i of radius η_i is included in $\text{Int}(k_i)$. Then, we define a function g in the following way:

For $0 \leq i \leq n + m - 2$, we put $g = b_{i+1}$ on k_i and $g = f$ on ∂K_i . On the one hand, g takes its values in K_{i+1} which is homeomorphic to I^N . On the other hand, $k_i \cup \partial K_i$ is a closed subset of K_i . So, according to the Tietze extension theorem (see [2, Corollary 1, p. 82]), g has a continuous extension (which means that it still agrees with f on ∂K_i and with b_{i+1} on k_i) defined on K_i and taking its values in K_{i+1} . As $d(K_{i+1}) < \varepsilon$, $d[f(x), g(x)] < \varepsilon$ for each $x \in K_i$.

In the same way, we put $g = b_n$ on k_{n+m-1} and $g = f$ on ∂K_{n+m-1} , and then we extend g on K_{n+m-1} into $\text{Clos}(V)$.

We put $g = f$ outside $\bigcup_i K_i$.

So, as g agrees with f on the boundaries of the K_i , it follows that g is

continuous on X . As $d[f(x), g(x)] < \varepsilon$ for each $x \in X$ and the function $x \rightarrow d[f(x), g(x)]$ is continuous on the compact X , we have $d_\infty(f, g) < \varepsilon$.

Let $b = b_0$ and $\eta = \min \eta_i$. As a and b are both in K_0 , $d(a, b) < \varepsilon$. Thus, we have defined $(g, b) \in C(X) \times X$ such that $\delta[(f, a), (g, b)] < \varepsilon$.

Let $(h, c) \in C(X) \times X$ be such that $\delta[(g, b), (h, c)] < \eta$, that is to say, $d_\infty(h, g) < \eta$ and $d(c, b) < \eta$.

If $x \in k_i$, then $g(x) = b_{i+1}$, so $d[b_{i+1}, h(x)] < \eta$; thus $d[b_{i+1}, h(x)] < \eta_{i+1}$, and so $h(x) \in k_{i+1}$. Therefore, $h(k_i) \subset k_{i+1}$ for each $0 \leq i \leq n+m-2$. For the same reason, $h(k_{n+m-1}) \subset k_n$.

As $c \in k_0$, for $l \geq n$, the sequence $(h^l(c))$ takes its values in $k_n \cup k_{n+1} \cup \dots \cup k_{n+m-1}$ which is a closed subset. Thus $\omega(h, c) \subset k_n \cup k_{n+1} \cup \dots \cup k_{n+m-1}$. As $m\rho^\alpha < 1$ and $\rho < \beta$, $d(k_i) < \beta$ for each i and $\sum_i d(k_i)^\alpha < 1$. Therefore, (h, c) belongs to $E_{\alpha, \beta}$. So the ball of $C(X) \times X$ centred at (g, b) of radius η is included in $E_{\alpha, \beta}$.

So any ball of $C(X) \times X$ (centred at (f, a) of radius ε) contains a point (g, b) which is the centre of a ball (of radius η) included in $E_{\alpha, \beta}$, and the result follows.

Lemma 2. Let U be an open set of X . Let F_U be the set of those $(f, x) \in C(X) \times X$ such that $\omega(f, x)$ either contains no point of U or contains at least two points of U . Then F_U contains an open, dense subset of $C(X) \times X$.

Proof. Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. We begin the construction as in the proof of Lemma 1. We obtain n and m as before. We shall distinguish two cases:

(1) For each i such that $n \leq i \leq n+m-1$, $f^i(a) \in \text{Ext}(U)$.

As $\text{Ext}(U)$ is open, we can choose the K_i for $0 \leq i \leq n+m-1$ such that $K_i \subset \text{Ext}(U)$ for $n \leq i \leq n+m-1$. We choose any $\rho > 0$ and then the k_i and the b_i . As in the proof of Lemma 1, we obtain $(g, b) \in C(X) \times X$ and $\eta > 0$ such that $\delta[(f, a), (g, b)] < \varepsilon$; and if $(h, c) \in C(X) \times X$ verifies $\delta[(g, b), (h, c)] < \eta$, then $\omega(h, c)$ has no point in U , so (h, c) belongs to F_U .

(2) There is a j such that $n \leq j \leq n+m-1$ and $f^j(a) \in \text{Clos}(U)$.

We can choose the K_i such that $K_j \cap U$ contains an open set W . Taking any $\rho > 0$, we chose the k_i and the b_i as in the proof of Lemma 1 for $i < n$. Then, for each $i \geq n$, we choose k_i and k'_i two disjoint compact subsets with non-empty interiors included in $\text{Int}(K_i)$, b_i and b'_i as in the proof of Lemma 1, and k_j and k'_j in W . Then we modify the construction of the function g in the following way:

- for $0 \leq i \leq n+m-2$, we put $g = b_{i+1}$ on k_i and $g = b'_{i+1}$ on k'_i ;
- we put $g = b'_n$ on k_{n+m-1} and $g = b_n$ on k'_{n+m-1} .

As in the proof of Lemma 1, we obtain $(g, b) \in C(X) \times X$ and $\eta > 0$ such that $\delta[(f, a), (g, b)] < \varepsilon$; and if $(h, c) \in C(X) \times X$ verifies $\delta[(g, b), (h, c)] < \eta$, then $\omega(h, c)$ has at least two points in U , so (h, c) belongs to F_U . The result follows.

RESULTS

Theorem 1. For (f, x) typical in $C(X) \times X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.

Proof. Let E be the intersection of the $E_{1/n, 1/m}$ for $n, m \geq 1$ and F_{U_p} , $(U_p)_p$ being a countable base of open subsets of X . As the set of the triples (n, m, p) is countable and according to Lemmas 1 and 2, E is residual.

Let $(f, x) \in E$. For each (α, β) , $(f, x) \in E_{\alpha, \beta}$; thus, $\omega(f, x)$ can be covered by a finite number of sets $(k_i)_i$ satisfying: for each i , $d(k_i) < \beta$ and $\sum_i d(k_i)^\alpha < 1$. Therefore, according to the definition of the Hausdorff dimension (see [2]), $\omega(f, x)$ is of Hausdorff dimension zero.

For each open subset U of X , $(f, x) \in F_U$; thus either $\omega(f, x)$ has no point in U or at least two. Therefore, $\omega(f, x)$ does not contain an isolated point. As $\omega(f, x)$ is compact, $\omega(f, x)$ is a perfect set.

So we have found E , a residual subset of $C(X) \times X$, such that if $(f, x) \in E$, then $\omega(f, x)$ is a perfect set of Hausdorff dimension zero. That is to say, for (f, x) typical in $C(X) \times X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.

Theorem 2. *If f is typical in $C(X)$, then, for x typical in X , $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.*

Proof. $C(X)$ and X are two complete metric spaces and X is second countable. Let E be the residual subset of $C(X) \times X$ of the proof of Theorem 1. According to the Kuratowski-Ulam theorem (see [4, p. 56]), there is a residual subset F of $C(X)$ such that for each $f \in F$ the section of E by f (i.e., $E_f = \{x \in X; (f, x) \in E\}$) is a residual subset of X . If $f \in F$ and $x \in E_f$, then $(f, x) \in E$, so $\omega(f, x)$ is a perfect set of Hausdorff dimension zero. That is to say, if f is typical in $C(X)$, then, for x typical in X , $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.

Theorem 3. *The function ω is continuous at the typical point (f, x) of $C(X) \times X$.*

Proof. Let $\alpha > 0$ and G_α be the set of those $(f, x) \in C(X) \times X$ such that there is a neighbourhood U of (f, x) on which $d_K[\omega(f, x), \omega(f', x')] < \alpha$ for each $(f', x') \in U$.

Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. We begin as in the proof of Lemma 1. We obtain the K_i and then choose $\rho > 0$ such that $m\rho < \alpha$. Then we obtain the k_i such that $d(k_n) + d(k_{n+1}) + \dots + d(k_{n+m-1}) < \alpha$. Then we construct $(g, b) \in C(X) \times X$ and $\eta > 0$ as in the proof of Lemma 1. Let B be the open ball of $C(X) \times X$ centred at (g, b) of radius η . If $(h, c) \in B$, then $\omega(h, c) \subset k_n \cup k_{n+1} \cup \dots \cup k_{n+m-1}$ and $\omega(h, c) \cap k_i \neq \emptyset$ for $n \leq i \leq n+m-1$. Thus, if (h, c) and (h', c') belong to B , then $d_K[\omega(h, c), \omega(h', c')] < \alpha$.

Let $(h, c) \in B$. Since B is an open set, there is an open neighbourhood V of (h, c) contained in B . If $(h', c') \in V$, then $d_K[\omega(h, c), \omega(h', c')] < \alpha$. Hence $(h, c) \in G_\alpha$. As in the proof of Lemma 1, it follows that G_α contains a dense, open subset of $C(X) \times X$.

The intersection G of the $G_{1/n}$ for $n \geq 1$ is residual. Let $(f, x) \in G$. Let $\alpha > 0$; then $(f, x) \in G_\alpha$, so there is a neighbourhood U of (f, x) such that $d_K[\omega(f, x), \omega(f', x')] < \alpha$ for each $(f', x') \in U$. This means that ω is continuous at the point (f, x) . Therefore, ω is continuous at any point of G . The result follows.

Theorem 4. *If f is typical in $C(X)$, the function $x \rightarrow \omega(f, x)$ is continuous at the typical point x of X .*

Proof. As in the proof of Theorem 2, the Kuratowski-Ulam theorem allows us to deduce this result from the previous one.

REMARKS

Remark 1. According to Mandelbrot (see [3, p. 15]), a fractal is a set for which the two dimensions used in this paper (and in [2]) are distinct. Hence the typical limit set is not a fractal.

Remark 2. In chaotic sequences, the limit set has a sensitive dependence on the initial condition. As for f typical in $C(X)$, the function $x \rightarrow \omega(f, x)$ is continuous at the typical point of X ; it follows that the sequence $(f^n(x))$ is not chaotic.

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