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OSCILLATORY SINGULAR INTEGRALS ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

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ABSTRACT. In this paper, it is proved that the oscillatory singular integral operators of nonconvolution type are bounded from Hardy spaces associated with Herz spaces to Herz spaces.

1. INTRODUCTION

Let T be an oscillatory singular integral operator defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy, \tag{1.1}$$

where $P(x, y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and K is a Calderòn-Zygmund kernel.

It is proved by D. H. Phong and E. M. Stein in [6] that T is a bounded operator from H_E^1 to L^1 provided $P(x, y)$ is a real bilinear form, where H_E^1 is certain variant of the H^1 space. Later, this result is extended into the case of general $P(x, y)$ by Y. B. Pan in [5]. For general $P(x, y)$, it is still an interesting problem whether T is a bounded operator from H^1 to L^1. Recently, some new Hardy spaces HK_p associated with Herz spaces K_p are introduced by the authors in [4] and [8]. The space HK_p is defined by

$$HK_p = \{f : Gf \in K_p\}, \tag{1.2}$$

where Gf is the Grand maximal function of f. An interesting fact shown in [8] is that HK_p is the localization of H^1 at the origin. It is easy to see that the relation between HK_p and K_p is similar to one between H^1 and L^1.

In this paper, we shall prove that T defined by (1.1) is a bounded operator from HK_p to K_p. A counterexample shows that there exists an operator T defined by (1.1), such that T is not a bounded operator from HK_p to itself. To formulate our result, let us first introduce some definitions.

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Definition 1.1 (see [3]). Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The Herz space $K_p(\mathbb{R}^n)$ consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n \backslash \{o\})$ for which

$$\|f\|_{K_p} := \sum_{k \in \mathbb{Z}} 2^{kn/p'} \|f\chi_k\|_p < \infty,$$

where $\chi_k = \chi_{C_k}$, $C_k = Q_k \backslash Q_{k-1}$, and $Q_k = \{x : |x| \leq 2^k\}$.

Definition 1.2 (see [4]). Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(1, p)$-atom if

(1) $\operatorname{Supp} a \subset Q$, where Q is a ball centered at the origin;
(2) $\|a\|_p \leq |Q|^{1/p-1}$;
(3) $\int a(x)\, dx = O$.

Now, we can state our result as follows.

Theorem. *Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $\nabla_y P(0, y) = O$, and T be defined as in (1.1). Then T maps $HK_p(\mathbb{R}^n)$ into $K_p(\mathbb{R}^n)$ and*

$$\|Tf\|_{K_p} \leq C\|f\|_{HK_p},$$

where C depends only on the total degree of $P(x, y)$ but not on the coefficients of $P(x, y)$.

2. Proof of the Theorem

To prove the Theorem, we need two lemmas.

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $f \in HK_p(\mathbb{R}^n)$ if and only if f can be represented as*

$$f(x) = \sum_i \lambda_i a_i(x),$$

where each a_i is a central $(1, p)$-atom and $\sum_i |\lambda_i| < \infty$. Moreover,

$$\|f\|_{HK_p} := \|Gf\|_{K_p} \sim \inf\left\{\sum_i |\lambda_i|\right\},$$

where the infimum is taken over all decompositions of f as above.

See [8] for the proof, and see [4] for other characterizations of HK_p.

The following lemma belongs to Y. B. Pan [5].

Lemma 2.2. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy*

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2, \end{cases}$$

and let $\psi \in C_o^\infty(\mathbb{R}^n)$ satisfy

$$\psi(x) = \begin{cases} 1 & \text{for } 1 \leq |x| \leq 2, \\ 0 & \text{for } |x| \leq \frac{1}{4} \text{ or } |x| \geq 4. \end{cases}$$

Define T_k by

$$T_k f(x) = \psi(x/2^k) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y)\, dy.$$

If $P(x, y)$ satisfies

$$P(x, y) = \sum_{|\alpha|\geq 1,\ |\beta|=l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),$$

where $Q(x, y)$ is a polynomial with degree in y less than or equal to $l-1$, then for each $N > 0$ (large enough) we have

$$\|T_k\|_{L^2\to L^2} \leq C2^{nk} |a_{\alpha_0\beta_0}|^{-1/2Nl} 2^{-k|\alpha_0|/2Nl},$$

where $|a_{\alpha_0\beta_0}| = \max_{|\alpha|\geq 1,\ |\beta|=l} |a_{\alpha\beta}|$.

Proposition 2.1. *Let $\delta > 0$. Then we have*

$$\|f(\delta\cdot)\|_{K_p} \sim \delta^{-n}\|f\|_{K_p}.$$

Proof. For any $\delta > 0$, there exists a $k_0 \in \mathbb{Z}$ such that $2^{k_0} < \delta \leq 2^{k_0+1}$. By Definition 1.1,

$$\begin{aligned}
\|f(\delta\cdot)\|_{K_p} &= \sum_{k\in\mathbb{Z}} 2^{kn/p'} \left(\int_{C_k} |f(\delta x)|^p\, dx\right)^{1/p} \\
&\leq \sum_{k\in\mathbb{Z}} 2^{kn/p'} \delta^{-n/p} \left(\int_{2^{k+k_0}<|y|\leq 2^{k+k_0+2}} |f(y)|^p\, dy\right)^{1/p} \\
&\leq \delta^{-n/p} 2^{-k_0 n/p'} \sum_{k\in\mathbb{Z}} 2^{(k+k_0)n/p'} \left(\int_{C_{k+k_0}} |f(y)|^p\, dy\right)^{1/p} \\
&\quad + \delta^{-n/p} 2^{-(k_0+1)n/p'} \sum_{k\in\mathbb{Z}} 2^{(k+k_0+1)n/p'} \left(\int_{C_{k+k_0+1}} |f(y)|^p\, dy\right)^{1/p} \\
&\leq C\delta^{-n}\|f\|_{K_p}.
\end{aligned}$$

On the other hand,

$$\|f\|_{K_p} = \|f(\delta^{-1}\delta\cdot)\|_{K_p} \leq C\delta^n \|f(\delta\cdot)\|_{K_p}.$$

This finishes the proof of Proposition 2.1.

By Lemma 2.1, it is easy to see that the proof of the Theorem is reduced to the following proposition.

Proposition 2.2. *Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial, $\nabla_y P(0, y) = 0$, and T be defined as in $(1, 1)$. Then for any central $(1, p)$-atom a,*

$$\|Ta\|_{K_p} \leq C,$$

where C is independent of a and the coefficients of $P(x, y)$.

Proof. Let $\operatorname{Supp} a \subset Q$ and Q be a ball centered at the origin with radius δ. If we write $b(x) = \delta^n a(\delta x)$, then $b(x)$ is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$. We also have

$$\begin{aligned}
Ta(\delta x) &= \delta^{-n} T_1 b(x) \\
&:= \delta^{-n}\, \text{p.v.} \int_{\mathbf{R}^n} e^{iP(\delta x, \delta y)} K_1(x-y) b(y)\, dy,
\end{aligned}$$

where $K_1(x) = \delta^n K(\delta x)$. By Proposition 2.1, we obtain

$$\|Ta\|_{K_p} \sim \|T_1 b\|_{K_p}.$$

Let $P_1(x, y) = P(\delta x, \delta y)$. Note that $\nabla_y P_1(0, y) = 0$ and K_1 is also a Calderón-Zygmund kernel. We may assume $T_1 = T$. Thus, it suffices to show

$$\|Tb\|_{K_p} \leq C, \tag{2.1}$$

where C is independent of b and the coefficients of $P(x, y)$ and b is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$.

We now turn to prove (2.1) by using induction on the degree of y, l, in $P(x, y)$. If $l = 0$, then

$$|Tb(x)| = \left|\text{p.v.} \int K(x, y) b(y)\, dy\right|.$$

Thus,

$$\begin{aligned}\|Tb\|_{K_p} &= \sum_{k\in\mathbb{Z}} 2^{kn/p'} \|(Tb)\chi_k\|_p \\ &= \sum_{k\leq 0} \cdots + \sum_{k>0} \cdots := S_1 + S_2.\end{aligned}$$

By L^p-boundedness of Calderón-Zygmund operators,

$$S_1 \leq C \sum_{k\leq 0} 2^{kn/p'} \|b\|_p = C \sum_{k\leq 0} 2^{kn/p'} = C.$$

From the condition of $K(x, y)$,

$$|K(x, y) - K(x, 0)| \leq C|y|/|x-y|^{n+1}, \quad \text{if } |y| < |x-y|/2,$$

it follows that

$$Tb(x) = \int b(y)[K(x, y) - K(x, 0)]\, dy$$

and

$$\begin{aligned}S_2 &= \sum_{k>0} 2^{kn/p'} \|(Tb)\chi_k\|_p \\ &\leq C \sum_{k>0} 2^{kn/p'} \left[\int_{C_k} \left(\int_{B(0,1)} \frac{|b(y)|\,|y|}{|x-y|^{n+1}}\, dy\right)^p dx\right]^{1/p} \\ &\leq C \sum_{k>0} 2^{kn/p'} \left(\int_{C_k} \frac{dx}{|x|^{(n+1)p}}\right)^{1/p} \|b\|_p \\ &\leq C \sum_{k>0} 2^{kn/p'} 2^{-k[(n+1)p-n]/p} = C \sum_{k>0} 2^{-k} = C.\end{aligned}$$

Therefore, (2.1) holds for $l = 0$. Let us now consider the case of $l > 0$. We assume that (2.1) holds for $l - 1$ by induction. Since $\nabla_y P(0, y) = 0$, we can write

$$P(x, y) = \sum_{|\alpha|\geq 1,\ |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),$$

where $Q(x, y)$ is a polynomial with degree in y less than or equal to $l - 1$ and $\nabla_y Q(0, y) = 0$. Denote

$$|a_{\alpha_0\beta_0}| = \max_{|\alpha|\geq 1,\ |\beta| = l} |a_{\alpha\beta}|$$

and

$$r = \max\{3, |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}\}. \tag{2.2}$$

Since $r \geq 3$, we may assume $2^{j_0} < r \leq 2^{j_0+1}$ for some $j_0 \in \mathbb{N}$. We now write

$$\begin{aligned} \|Tb\|_{K_p} &= \sum_{j\leq 0} 2^{jn/p'}\|(Tb)\chi_j\|_p + \sum_{j=1}^{j_0} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &\quad + \sum_{j\geq j_0+1} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

By L^p-boundedness of oscillatory singular integral operators (see [7]), we have

$$I_1 \leq C\sum_{j\leq 0} 2^{jn/p'}\|b\|_p = C\sum_{j\leq 0} 2^{jn/p'} = C.$$

To estimate I_2, we may assume $j_0 \geq 2$. In this case, $r = |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$. By induction assumption,

$$\begin{aligned} I_2 &= \sum_{j=1}^{j_0} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &\leq \sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left|\int_{\mathbb{R}^n}(e^{iP(x,y)} - e^{iQ(x,y)})K(x-y)b(y)\,dy\right|^p dx\right\}^{1/p} \\ &\quad + \sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left|\int_{\mathbb{R}^n} e^{iQ(x,y)}K(x-y)b(y)\,dy\right|^p dx\right\}^{1/p} \\ &\leq C\sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left[\int_{|y|\leq 1}\left|\exp\left(i\sum_{|\alpha|\geq 1,\ |\beta|=l} a_{\alpha\beta}x^\alpha y^\beta\right)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. - 1\right|\frac{|b(y)|}{|x|^n}\,dy\right]^p dx\right\}^{1/p} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|\sum_{j=1}^{j_0} 2^{jn/p'}\left(\int_{C_j}|x|^{(|\alpha|-n)p}\,dx\right)^{1/p} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|\sum_{j=1}^{j_0} 2^{j|\alpha|} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|r^{|\alpha|} + C \\ &\leq C_1|a_{\alpha_0\beta_0}|r^{|\alpha_0|} + C = C_1 + C. \end{aligned}$$

It remains to estimate I_3. Let φ and ψ be the functions as in Lemma 2.2.

Then

$$\begin{aligned}
I_3 &= \sum_{j\ge j_0+1} 2^{jn/p'}\|(Tb)\chi_j\|_p \\
&\le \sum_{j\ge j_0+1} 2^{jn/p'}\left\{\int_{C_j}\left(\int_{\mathbb{R}^n}|K(x-y)-K(x)|\,|b(y)|\,dy\right)^p dx\right\}^{1/p} \\
&\quad + \sum_{j\ge j_0+1} 2^{jn/p'}\left\{\int_{C_j}\frac{1}{|x|^{np}}\left|\int_{\mathbb{R}^n} e^{iP(x,y)}b(y)\,dy\right|^p dx\right\}^{1/p} \\
&\le \sum_{j\ge j_0+1} 2^{jn/p'}\left(\int_{C_j}\frac{dx}{|x|^{(n+1)p}}\right)^{1/p} + \sum_{j\ge j_0+1} 2^{-jn/p}\|T_jb\|_p \\
&\le C + \sum_{j\ge j_0+1} 2^{-jn/p}\|T_jb\|_p.
\end{aligned}$$

By Lemma 2.2, we have

$$\|T_jb\|_2 \le C2^{jn/2}|a_{\alpha_0\beta_0}|^{-1/2Nl}2^{-j|\alpha_0|/2Nl}\|b\|_2.$$

It is easy to check from the definition of T_j that

$$\|T_jb\|_\infty \le C\|b\|_\infty$$

and

$$\|T_jb\|_1 \le C2^{jn}\|b\|_1.$$

By the interpolation theorem, we obtain

$$\|T_jb\|_p \le \begin{cases} C2^{jn/p}|a_{\alpha_0\beta_0}|^{-1/Nlp'}2^{-j|\alpha_0|/Nlp'}\|b\|_p & \text{for } 1 < p \le 2, \\ C2^{jn/p}|a_{\alpha_0\beta_0}|^{-1/Nlp}2^{-j|\alpha_0|/Nlp}\|b\|_p & \text{for } 2 < p < \infty. \end{cases}$$

It follows from the above and (2.2) that if $1 < p \le 2$, then

$$\begin{aligned}
I_3 &\le C + C|a_{\alpha_0\beta_0}|^{-1/Nlp'}\sum_{j\ge j_0+1} 2^{-j|\alpha_0|/Nlp'} \\
&\le C + C(|a_{\alpha_0\beta_0}|r^{|\alpha_0|})^{-1/Nlp'} \le C;
\end{aligned}$$

and if $2 < p < \infty$, then

$$\begin{aligned}
I_3 &\le C + C|a_{\alpha_0\beta_0}|^{-1/Nlp}\sum_{j\ge j_0+1} 2^{-j|\alpha_0|/Nlp} \\
&\le C + C(|a_{\alpha_0\beta_0}|r^{|\alpha_0|})^{-1/Nlp} \le C.
\end{aligned}$$

This completes the proof of (2.1) and therefore the proof of Proposition 2.2.

Remark 2.1. Recently, Hardy spaces $HA^p(\mathbb{R}^n)$ related to the Beurling algebras $A^p(\mathbb{R}^n)$ have been introduced by Y. Z. Chen and K. S. Lau in [1] and independently by J. Garcia-Cuerva in [2]. It has been proved by the authors in [4] that

$$HK_p \cap L^p = HA^p$$

and

$$\|f\|_{HA^p} \sim \|f\|_{HK_p} + \|f\|_p. \tag{2.3}$$

On the other hand, it is easy to show that

$$\|f\|_{A^p} \sim \|f\|_{K_p} + \|f\|_p. \tag{2.4}$$

Thus, from (2.3), (2.4), and the Theorem, it is easy to see that under the conditions of Theorem, T defined by (1.1) is a bounded operator from $HA^p(\mathbb{R}^n)$ to $A^p(\mathbb{R}^n)$ and

$$\|Tf\|_{A^p} \leq C\|f\|_{HA^p}.$$

Remark 2.2. A counterexample shows that there exists an operator T defined by (1.1) such that T is not a bounded operator from HK_p to itself. Let us consider $n = 1$. Take a $g \in HK_p(\mathbb{R})$ such that $Hg(x) \neq 0$ a.e., where Hg is the Hilbert transform of g. Let $P(x, y) = tx$, $t \in \mathbb{R}$. Suppose T is a bounded operator from HK_p into itself. Then $Tg \in HK_p(\mathbb{R})$. Thus, by Lemma 2.1, we have

$$\int Tg(x)\,dx = 0.$$

This is

$$\int e^{itx} Hg(x)\,dx = 0, \qquad t \in \mathbb{R}.$$

Hence, $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$. It has been proved for the case of $l = 0$ in the proof of Theorem that H maps HK_p into K_p. Thus, $Hg \in K_p \subset L^1$. Combining it with $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$, we get a contradiction,

$$Hg(x) = 0 \quad \text{a.e.}$$

This confirms the above assertion. However, for the oscillatory integral operator T of convolution type with $P(x, y) = P(x - y)$, the second-named author has proved that T maps HK_p into itself provided $\nabla P(0) = 0$. We omit it here.

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