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$\mathbb{Q}(t)$ AND $\mathbb{Q}((t))$-ADMISSIBILITY OF GROUPS OF ODD ORDER

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ABSTRACT. Let $\mathbb{Q}(t)$ be the rational function field over the rationals, $\mathbb{Q}$, let $\mathbb{Q}((t))$ be the Laurent series field over $\mathbb{Q}$, and let $\mathcal{G}$ be a group of odd order. We investigate the following question: does there exist a finite-dimensional division algebra D central over $\mathbb{Q}(t)$ or $\mathbb{Q}((t))$ which is a crossed product for $\mathcal{G}$? If such a D exists, $\mathcal{G}$ is said to be $\mathbb{Q}(t)$-admissible (respectively, $\mathbb{Q}((t))$-admissible). We prove that if $\mathcal{G}$ is $\mathbb{Q}((t))$-admissible, then $\mathcal{G}$ is also $\mathbb{Q}(t)$-admissible; we also exhibit a $\mathbb{Q}(t)$-admissible group which is not $\mathbb{Q}((t))$-admissible.

Let K be a field and let $\mathcal{G}$ be a finite group. $\mathcal{G}$ is said to be K-admissible if there exists a division algebra D, finite dimensional and central over K, which is a crossed product for $\mathcal{G}$. Equivalently, $\mathcal{G}$ is K-admissible if there exists a division algebra D with center K having a maximal subfield L Galois over K with $\mathrm{Gal}(L/K) \cong \mathcal{G}$. Admissibility questions for $K = \mathbb{Q}$, the field of rational numbers, have been studied extensively in the literature (e.g., [Sc] and [So_2]). More recently, results have been obtained when K is an algebraic function field over some field K_0 ([FSS] and [FS]). In this paper we study admissibility questions for groups of odd order when K is either the rational function field $\mathbb{Q}(t)$ or the Laurent series field $\mathbb{Q}((t))$. We show for such groups that $\mathbb{Q}((t))$-admissibility implies $\mathbb{Q}(t)$-admissibility but not conversely. We also construct examples of groups of odd order which are $\mathbb{Q}((t))$-admissible but which have homomorphic images which are not $\mathbb{Q}((t))$-admissible; by contrast, if K is a number field, a homomorphic image of a K-admissible group is necessarily K-admissible [Sc, Corollary 2.3].

We fix below most of the basic terminology and notation that we will employ throughout this paper. Let K be a field. By a K-division algebra we mean a division algebra having center K which is finite dimensional over K. We say that A/K is central simple if A is a simple algebra with center K which is finite dimensional over K. Suppose A/K is central simple. By Wedderburn's Theorem, $A \cong M_n(D)$ where D is a K-division algebra; we refer to D as the division algebra component of A. The Schur index of A, $\mathrm{ind}(A)$, equals

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$\sqrt{[D:K]}$. If p is a prime, A can be uniquely expressed in the form $A = A_p \otimes_K B$ where A_p/K and B/K are central simple, $[A:K]$ is a power of p, and $[B:K]$ is prime to p. The class of A in the Brauer group, $\mathrm{Br}(K)$, of K will be denoted $[A]$. If $\alpha \in \mathrm{Br}(K)$, we define the Schur index of α, $\mathrm{ind}(\alpha)$, to be $\mathrm{ind}(B)$ for any B/K central simple with $\alpha = [B]$. The order of α in $\mathrm{Br}(K)$ is denoted $\exp(\alpha)$. We say that E/K is $\mathscr{G}$-Galois if $E \supseteq K$ is a finite Galois extension of K with $\mathrm{Gal}(E/K) \cong \mathscr{G}$. Suppose that E/K is $\mathscr{G}$-Galois with $\mathscr{G}$ cyclic, generated by σ. If $b \in K^*$, we let $(E/K, \sigma, b)$ denote the cyclic crossed product algebra generated over E by an element x with defining relations $xex^{-1} = \sigma(e)$ for all $e \in E$ and $x^{|\mathscr{G}|} = b$.

Now let L be an extension field of K and let $\alpha = [A] \in \mathrm{Br}(K)$. We say that L splits A (or that L splits $[A]$) if $[A \otimes_K L] = 0 \in \mathrm{Br}(L)$; the subgroup of $\mathrm{Br}(K)$ consisting of thoses classes split by L is the relative Brauer group, $\mathrm{Br}(L/K)$, of L over K. If L is a finite extension of K, we denote the corestriction homomorphism from $\mathrm{Br}(L)$ to $\mathrm{Br}(K)$ by cor_K^L.

Suppose D is a K-division algebra and E/K is a finite extension of fields. We will use freely the following two basic facts: if E splits D, then $\mathrm{ind}(D)$ divides $[E:K]$ [P, Proposition 13.4(v)]; if $[E:K] = \mathrm{ind}(D)$, then E is a maximal subfield of D if and only if E splits D [P, Corollary 13.3].

We assume that the reader is familiar with the basic results of Albert, Brauer, Hasse, and Noether which classify division algebras over number fields K; an exposition of the relevant theory may be found, for example, in [P, Chapter 18]. We denote the Hasse invariant of a central simple K-algebra A at a prime π of K by $\mathrm{inv}_\pi(A)$. The denominator of $\mathrm{inv}_\pi(A)$ (viewing $\mathrm{inv}_\pi(A)$ as a fraction in lowest terms) will be referred to as the local index of A at π; the local index of A at π equals $\mathrm{ind}(A \otimes_K K_\pi)$. We will freely use the following standard results of this theory. Suppose A/K is central simple. Then $\mathrm{inv}_\pi(A) \neq 0$ for only finitely many primes π of K, say for $\{\pi_1, \dots, \pi_n\}$. Let $\mathrm{inv}_{\pi_i}(A) = a_i/b_i$ where $a_i, b_i \in \mathbb{Z}$, $b_i > 0$, and $(a_i, b_i) = 1$. Then $\sum_i \frac{a_i}{b_i} \in \mathbb{Z}$ and $\mathrm{ind}(D) = \exp(D)$ equals the least common multiple of the b_i's [P, Theorem 18.6 and Corollary 18.6]. In particular, if $\mathrm{ind}(A) = p^r$ where p is prime, there must exist two primes π of K for which the local index of A at π equals p^r. If L is a finite extension of K and δ is a prime of L extending a prime π of K, then $\mathrm{inv}_\delta(A \otimes_K L) = [L_\delta : K_\pi] \cdot \mathrm{inv}_\pi(A)$ [P, Lemma 18.4]. In particular, L splits A if and only if, for every prime π of K and every extension δ of π to L, the local index of A at π divides $[L_\delta : K_\pi]$ [P, Corollary 18.4b]. Finally, suppose $\pi_1, \dots, \pi_n$ is a set of primes of K and a_i/b_i, $i = 1, \dots, n$, are given which satisfy: $a_i, b_i \in \mathbb{Z}$, $b_i > 0$, $(a_i, b_i) = 1$, $a_i/b_1 = 1/2$ if π_i is real infinite, $a_i/b_1 = 0$ if π_i is complex infinite, and $\sum_i \frac{a_i}{b_i} \in \mathbb{Z}$. Then there exists a unique K-division algebra D such that $\mathrm{inv}_{\pi_i}(D) = a_i/b_i$ for $i = 1, \dots, n$ and $\mathrm{inv}_\gamma(D) = 0$ for all other primes γ of K [P, Theorem 18.5].

We will need to use several results from the theory of division algebras over complete fields; we refer the reader to [JW] and [Se_1] as general references. We briefly summarize the results that we will be using. Suppose that K is complete with respect to a discrete rank-one valuation π. Let $\bar{K}$ be the residue field and assume that $\bar{K}$ is perfect. If $\bar{L}$ is a finite extension of $\bar{K}$, there exists a unique unramified extension L of K of degree $[\bar{L}:\bar{K}]$ whose residue field is $\bar{L}$; we refer to L as the *inertial lift* of $\bar{L}$. We call an element $t \in K$ a uniformizing element if t generates the maximal ideal of the ring of π-integers. Let D be a K-division algebra. The valuation on K extends uniquely to a valuation on D. We let $\bar{D}$ denote the residue division algebra. If $\bar{B}$ is a $\bar{K}$-division algebra, there is a unique K-division algebra D with $[D : K] = [\bar{B} : \bar{K}]$ and with $\bar{D} = \bar{B}$. We refer to D as the *inertial lift* of $\bar{B}$. The existence of D follows, for example, from [JW, Theorem 2.8]. We will make frequent use of the following fundamental result:

Proposition 1. *Let K be a field complete with respect to a discrete rank-one valuation having a perfect residue field $\bar{K}$. Let $t \in K$ be a uniformizing element and let $\alpha \in \mathrm{Br}(K)$.*

(1) *There exists a unique $\bar{K}$-division algebra $\bar{D}$, a unique cyclic extension $\bar{L}$ of $\bar{K}$, and a unique generator $\bar{\sigma}$ for $\mathrm{Gal}(\bar{L}/\bar{K})$ such that $\alpha = [D \otimes_K (L/K, \sigma, t)]$; here D is the inertial lift of $\bar{D}$, L is the inertial lift of $\bar{L}$, and σ is the generator corresponding to $\bar{\sigma}$ in the canonical isomorphism between $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(\bar{L}/\bar{K})$.*

Let $\bar{D}$ and $\bar{L}$ be as in (1).

(2) *Let $\bar{E}$ be a finite extension of $\bar{K}$ and let E be the inertial lift of $\bar{E}$. Then E splits α if and only if $\bar{E}$ splits $\bar{D}$ and $\bar{E} \supseteq \bar{L}$.*

(3) $\mathrm{ind}(\alpha) = [\bar{L} : \bar{K}] \cdot \mathrm{ind}(\bar{D} \otimes_{\bar{K}} \bar{L})$.

Proof. For the first assertion, see [Se_1, Chapter 12, Theorem 2]; for the second, see [Se_1, Chapter 12, Exercise 2]. The third assertion follows from [JW, Theorem 5.15]; an elementary proof appears in [FSS, Lemma 4.6].

In our applications of Proposition 1, K will always be $\mathbb{Q}((t))$ and $\bar{K}$ will be $\mathbb{Q}$. If $L/\mathbb{Q}$ is $\mathscr{G}$-Galois, we identify $\mathscr{G}$ as $\mathrm{Gal}(L((t))/\mathbb{Q}((t)))$ by letting $\mathscr{G}$ act trivially on t. We begin our study of $\mathbb{Q}((t))$-admissibility with a preliminary result.

Lemma 2. *Let $\widehat{E}/\mathbb{Q}((t))$ be a $\mathscr{G}$-Galois extension of odd degree. Then there exists a $\mathscr{G}$-Galois extension E of $\mathbb{Q}$ such that $\widehat{E} = E((t))$.*

Proof. Let $\widehat{T}$ be the maximal unramified extension of $\mathbb{Q}((t))$ in $\widehat{E}$. Then $\widehat{E}/\widehat{T}$ is a totally and tamely ramified extension. By [W, Proposition 3-4-3], $\widehat{E} = \widehat{T}(\sqrt[e]{\pi})$ where π is a prime element of $\widehat{T}$ and $e = [\widehat{E} : \widehat{T}]$. Since $\widehat{E}/\widehat{T}$ is Galois, $\widehat{T}$ contains a primitive e-th root of unity, ζ. Since $[\widehat{T} : \mathbb{Q}((t))]$ is odd, $[\mathbb{Q}((t))(\zeta) : \mathbb{Q}((t))]$ is odd. Since e is odd, $e = 1$, and so $\widehat{E}/\mathbb{Q}((t))$ is unramified. Thus $\widehat{E} = E((t))$ where $E/\mathbb{Q}$ is $\mathscr{G}$-Galois. □

Theorem 3. *Let $\mathscr{G}$ be a $\mathbb{Q}((t))$-admissible group of odd order. Then $\mathscr{G}$ is $\mathbb{Q}(t)$-admissible.*

Proof. By assumption, there exists a $\mathbb{Q}((t))$-division algebra D having a $\mathscr{G}$-Galois maximal subfield. Since $|\mathscr{G}|$ is odd, Lemma 2 implies that this maximal subfield is of the form $E((t))$ where $E/\mathbb{Q}$ is $\mathscr{G}$-Galois. By Proposition 1(1), there exists a $\mathbb{Q}$-division algebra D_0, a cyclic extension L of $\mathbb{Q}$, and a generating automorphism σ for $\mathrm{Gal}(L/\mathbb{Q})$ such that $[D] = [D_0 \otimes_{\mathbb{Q}} \mathbb{Q}((t))] + [(L((t))/\mathbb{Q}((t)), \sigma, t)]$. Since $E((t))$ splits D, E splits D_0 and $E \supseteq L$ by Proposition 1(2). Let D_1 be the division algebra component of $[D_0 \otimes_{\mathbb{Q}} \mathbb{Q}(t)] + [(L(t)/\mathbb{Q}(t), \sigma, t)] \in \mathrm{Br}(\mathbb{Q}(t))$. Since E splits D_0 and $E \supseteq L$,

$E(t)$ splits D_1, and so $\operatorname{ind}(D_1) \leq [E(t):\mathbb{Q}(t)] = |\mathscr{G}|$. But $[D_1 \otimes_{\mathbb{Q}(t)} \mathbb{Q}((t))] = [D]$, and so $\operatorname{ind}(D_1) \geq \operatorname{ind}(D) = |\mathscr{G}|$. Thus $\operatorname{ind}(D_1) = |\mathscr{G}| = [E(t):\mathbb{Q}(t)]$. Since $E(t)$ splits D_1 and $[E(t):\mathbb{Q}(t)] = \operatorname{ind}(D_1)$, $E(t)$ is a $\mathscr{G}$-Galois maximal subfield of D_1. Thus $\mathscr{G}$ is $\mathbb{Q}(t)$-admissible. □

We will show that the converse of Theorem 3 does not hold. We first exhibit a class of groups of odd order which are $\mathbb{Q}(t)$-admissible and then show that certain groups in this class are not $\mathbb{Q}((t))$-admissible.

Definition. A finite group $\mathscr{G}$ is said to be *meta-cyclic* if $\mathscr{G}$ has a cyclic normal subgroup with cyclic quotient group.

In the above definition, cyclic groups are considered to be meta-cyclic.

Theorem 4. *Let t be transcendental over $\mathbb{Q}$ and let $\mathscr{G}$ be a group of odd order. Assume that for every Sylow subgroup $\mathscr{P}$ of $\mathscr{G}$, there exists $\mathscr{P}_0 \lhd \mathscr{P}$ with $\mathscr{P}/\mathscr{P}_0$ cyclic such that either:*

(1) *$\mathscr{P}_0$ is meta-cyclic, or*

(2) *$\mathscr{P}_0$ can be generated by two elements and $[\mathscr{P}:\mathscr{P}_0] \geq |\mathscr{P}_0|$.*

Then $\mathscr{G}$ is $\mathbb{Q}(t)$-admissible.

Proof. Let $|\mathscr{G}| = p_1^{a_1} \dots p_r^{a_r}$ where the p_i's are distinct primes. For each $i = 1, \dots, r$, fix a Sylow p_i-subgroup $\mathscr{P}_i$ of $\mathscr{G}$. By assumption, there exists $\bar{\mathscr{P}}_i \lhd \mathscr{P}_i$ with $\mathscr{P}_i/\bar{\mathscr{P}}_i$ cyclic such that $\bar{\mathscr{P}}_i$ can be generated by two elements; if $\bar{\mathscr{P}}_i$ is not meta-cyclic, then $\bar{\mathscr{P}}_i$ also satisfies $[\mathscr{P}_i:\bar{\mathscr{P}}_i] \geq |\bar{\mathscr{P}}_i|$. For $1 \leq i \leq r$, $\bar{\mathscr{P}}_i$ is a Galois group over $\mathbb{Q}_{p_i}$ by [Se$_2$, Chapter 2, Section 5.6, Theorem 3]. Let $\mathscr{M} = \{i \mid \bar{\mathscr{P}}_i \text{ is meta-cyclic}\}$ and let $\mathscr{M}' = \{i \mid 1 \leq i \leq r\} - \mathscr{M}$. By [So$_1$, Theorem 1], there exists a set $\{q_i \mid i \in \mathscr{M}\}$ of distinct rational primes with no q_i in $\{p_1, \dots, p_r\}$ such that, for each i with $\bar{\mathscr{P}}_i$ meta-cyclic, $\bar{\mathscr{P}}_i$ is a Galois group over $\mathbb{Q}_{q_i}$. By [N, Corollary 2], there exists a $\mathscr{G}$-Galois extension $E/\mathbb{Q}$ such that $\bar{\mathscr{P}}_i$ is a decomposition group for $E/\mathbb{Q}$ at p_i for $1 \leq i \leq r$ and also at q_i if $\bar{\mathscr{P}}_i$ is meta-cyclic.

For $1 \leq i \leq r$, let $|\bar{\mathscr{P}}_i| = p_i^{b_i}$, let L_i be the fixed field of $\bar{\mathscr{P}}_i$, let K_i be the fixed field of $\mathscr{P}_i$, and let $c_i = a_i - b_i$. Then L_i/K_i is cyclic of degree $p_i^{c_i}$. We note that our hypotheses imply that $c_i \geq b_i$ if $\bar{\mathscr{P}}_i$ is not meta-cyclic. Let σ_i be a generator for $\operatorname{Gal}(L_i/K_i)$ and let $\hat{\sigma}_i \in \mathscr{G}$ extend σ. By the Tchebotarev Density Theorem [P, Theorem 18.7], there exists a set $\{t_i \mid i \in \mathscr{M}'\}$ of distinct rational primes with no t_i in $\{p_1, \dots, p_r\} \cup \{q_i \mid i \in \mathscr{M}\}$ such that $\langle \hat{\sigma}_i \rangle$ is a decomposition group for E over $\mathbb{Q}$ at t_i. For $1 \leq i \leq r$, let D_i be the $\mathbb{Q}$-division algebra such that:

(1) $\operatorname{inv}_{p_i}([D_i]) = p_i^{-b_i}$;

(2) if $\bar{\mathscr{P}}_i$ is meta-cyclic, then $\operatorname{inv}_{q_i}([D_i]) = -p_i^{-b_i}$ and $\operatorname{inv}_w([D_i]) = 0$ if $w \notin \{p_i, q_i\}$;

(3) if $\bar{\mathscr{P}}_i$ is not meta-cyclic, then $\operatorname{inv}_{t_i}([D_i]) = -p_i^{-b_i}$ and $\operatorname{inv}_w([D_i]) = 0$ if $w \notin \{p_i, t_i\}$.

The form of the Hasse invariants for $[D_i]$ imply that $\operatorname{ind}(D_i) = p_i^{b_i}$. Since $E/\mathbb{Q}$ is Galois, all primes of E extending a given prime of $\mathbb{Q}$ have the same local degree over $\mathbb{Q}$. By our choice of E, all primes of E lying over p_i have local degree $p_i^{b_i}$ over $\mathbb{Q}$. If $\bar{\mathscr{P}}_i$ is meta-cyclic, all primes of E lying over q_i

also have local degree $p_i^{b_i}$ over $\mathbb{Q}$; if $\bar{\mathscr{P}}_i$ is not meta-cyclic, all primes of E lying over t_i have local degree $|\langle\hat{\sigma}_i\rangle| \geq |\langle\sigma\rangle| = p_i^{c_i} \geq p_i^{b_i}$ over $\mathbb{Q}$. It follows that E splits D_i. Moreover, since p_i has an extension of degree one to L_i, $\operatorname{ind}(D_i \otimes_{\mathbb{Q}} L_i) = p_i^{b_i}$.

By [FSS, Lemma 4.7], there exists $d_i \in K_i(t)^*$ such that if $\alpha_i = [D_i \otimes_{\mathbb{Q}} \mathbb{Q}(t)] + \beta_i$ where $\beta_i = \operatorname{cor}_{\mathbb{Q}(t)}^{K_i(t)}([(L_i(t)/K_i(t), \sigma_i, d_i)])$, then $\operatorname{ind}(\alpha_i) \geq \operatorname{ind}(D_i \otimes_{\mathbb{Q}} L_i) \cdot [\mathscr{P}_i : \bar{\mathscr{P}}_i] = p^{b_i + c_i} = |\mathscr{P}_i|$. We will show that $\operatorname{ind}(\alpha_i) = |\mathscr{P}_i|$ and that $E(t)$ splits α_i.

Since $E(t) \supseteq L_i(t)$, $\beta_i \in \operatorname{Br}(E(t)/K_i(t)) \cong H^2(\mathscr{P}_i, E(t)^*)$. The corestriction map $\operatorname{cor}_{\mathbb{Q}(t)}^{K_i(t)} : \operatorname{Br}(K_i(t)) \longrightarrow \operatorname{Br}(\mathbb{Q}(t))$ corresponds to the cohomological corestriction map $\operatorname{cor}_{\mathscr{P}_i}^{\mathscr{G}} : H^2(\mathscr{P}_i, E(t)^*) \longrightarrow H^2(\mathscr{G}, E(t)^*) \cong \operatorname{Br}(E(t)/\mathbb{Q}(t))$. It follows that β_i is split by $E(t)$. Since E splits D_i, $E(t)$ splits α_i. Moreover, since $[L_i(t) : K_i(t)]$ is a power of p_i, $\operatorname{ind}(\beta_i)$ is a power of p and so $\exp(\beta_i)$ is a power of p [P, Proposition 14.4b(ii)]. Since $\operatorname{cor}_{\mathbb{Q}(t)}^{K_i(t)}$ is a homomorphism, $\exp(\operatorname{cor}_{\mathbb{Q}(t)}^{K_i(t)}(\beta_i))$ is a power of p_i. Since $\exp(D_i)$ is also a power of p_i, $\exp(\alpha_i)$ is a power of p_i and so $\operatorname{ind}(\alpha_i)$ is a power of p_i [P, Proposition 14.4b(ii)]. Since $|\mathscr{P}_i|$ is the exact power of p dividing $[E(t) : \mathbb{Q}(t)]$ and $\operatorname{ind}(\alpha_i) \geq |\mathscr{P}_i|$, it follows that $\operatorname{ind}(\alpha_i) = |\mathscr{P}_i|$.

Let D be the division algebra component of $\bigotimes_{i=1}^{r} \alpha_i$, the tensor product being taken over $\mathbb{Q}(t)$. Then $\operatorname{ind}(D) = |\mathscr{G}|$ [P, Proposition 14.4b(viii)]. Since $E(t)$ splits D and $[E(t) : \mathbb{Q}(t)] = \operatorname{ind}(D)$, $E(t)$ is a maximal subfield of D and so $\mathscr{G}$ is $\mathbb{Q}(t)$-admissible. $\square$

For future reference, we note that the proof of Theorem 4 yields a $\mathbb{Q}((t))$-admissibility result for groups of a very special type. Suppose that p is an odd prime, $\mathscr{H}$ is a p-group that can be generated by two elements, and $\mathscr{T}$ is a cyclic p-group with $|\mathscr{T}| \geq |\mathscr{H}|$. Let $\mathscr{P} = \mathscr{H} \times \mathscr{T}$. Following the proof of Theorem 4, we construct a $\mathscr{P}$-Galois extension $E/\mathbb{Q}$ such that $\mathscr{H}$ is a decomposition group at p and we construct a $\mathbb{Q}$-division algebra D of Schur index $|\mathscr{H}|$ split by E and such that $\operatorname{ind}(D \otimes_{\mathbb{Q}} E^{\mathscr{H}}) = |\mathscr{H}|$. Let $\hat{D} = D((t)) \otimes_{\mathbb{Q}((t))} (E^{\mathscr{H}}((t))/\mathbb{Q}((t)), \sigma, t)$ where $D((t))$ denotes the inertial lift of D to $\mathbb{Q}((t))$. By Proposition 1(3), $\operatorname{ind}(\hat{D}) = |\mathscr{P}| = [\hat{D} : \mathbb{Q}((t))]^{1/2}$. It follows that $\hat{D}$ is a $\mathbb{Q}((t))$-division algebra. Since $E((t))$ splits $\hat{D}$, $E((t))$ is a maximal subfield of $\hat{D}$. This proves:

Corollary 5. *Let p be an odd prime, let $\mathscr{H}$ be a p-group that can be generated by two elements, and let $\mathscr{T}$ be a cyclic p-group with $|\mathscr{T}| \geq |\mathscr{H}|$. Then $\mathscr{H} \times \mathscr{T}$ is $\mathbb{Q}((t))$-admissible.*

Lemma 6. *Let $\mathscr{G}$ be a group of odd order for which there exists a prime p satisfying:*

(1) *$\mathscr{G}$ has a Sylow p-subgroup which is not meta-cyclic, and*

(2) *$\mathscr{G}$ has no homomorphic image of order p.*

Then $\mathscr{G}$ is not $\mathbb{Q}((t))$-admissible.

Proof. Assume that $\mathscr{G}$ is $\mathbb{Q}((t))$-admissible and let $\widehat{D}$ be a $\mathbb{Q}((t))$-division algebra which is a crossed product for $\mathscr{G}$. Let $\widehat{E}$ be a $\mathscr{G}$-Galois maximal subfield of $\widehat{D}$. By Lemma 2, $\widehat{E} = E((t))$ where $E/\mathbb{Q}$ is $\mathscr{G}$-Galois. By Proposition 1(1), there exists a $\mathbb{Q}$-division algebra D, a cyclic extension $L/\mathbb{Q}$, and a generator σ for $\mathrm{Gal}(L/\mathbb{Q})$ such that $[\widehat{D}] = [D \otimes_{\mathbb{Q}} \mathbb{Q}((t))] + [(L((t))/\mathbb{Q}((t)), \sigma, t)]$. Since $\widehat{E} = E((t))$ splits $\widehat{D}$, E splits D and $E \supseteq L$. Since $\mathscr{G}$ has no cyclic quotient of order p, p does not divide $[L : \mathbb{Q}] = [L((t)) : \mathbb{Q}((t))]$. Since $L((t))$ splits $(L((t))/\mathbb{Q}((t)), \sigma, t)$, $\mathrm{ind}((L((t))/\mathbb{Q}((t)), \sigma, t))$ is prime to p. It follows that $\widehat{D}_p = D_p \otimes_{\mathbb{Q}} \mathbb{Q}((t))$. Let $|\mathscr{G}| = p^r m$ where $(p, m) = 1$. Since $\mathrm{ind}(\widehat{D}) = |\mathscr{G}|$, $\mathrm{ind}(D_p) = p^r$. It follows that there exists a prime q of $\mathbb{Q}$, $q \neq p$, such that D_p has local index p^r at q. Let π be an extension of q to E. Since E splits D, E splits D_p. Thus p^r divides $[E_\pi : \mathbb{Q}_q]$. Since $\mathrm{Gal}(E_\pi/\mathbb{Q}_q) \subseteq \mathscr{G}$, there exists a Sylow p-subgroup $\mathscr{P}$ of $\mathscr{G}$ with $\mathscr{P} \subseteq \mathrm{Gal}(E_\pi/\mathbb{Q}_q)$. Let $V \subset E_\pi$ be the fixed field of $\mathscr{P}$. Since $q \neq p$, E_π/V is tamely ramified and so $\mathscr{P} = \mathrm{Gal}(E_\pi/V)$ is meta-cyclic [W, Theorem 3-5-3 and Proposition 3-6-4], contrary to (1). Thus $\mathscr{G}$ is not $\mathbb{Q}((t))$-admissible. □

Corollary 7. *There exists a $\mathbb{Q}(t)$-admissible group which is not $\mathbb{Q}((t))$-admissible.*

Proof. Let $\mathscr{G}$ be the group with generators x_1, x_2, x_3, y, z and relations: $x_1^7 = x_2^7 = x_3^7 = 1$, $y^3 = z^3 = 1$, $yz = zy$, $x_i x_j = x_j x_i$ for $1 \le i, j \le 3$, $y x_i y^{-1} = x_{i+1}$ for $i = 1, 2$, $y x_3 y^{-1} = x_1$, and $z x_i z^{-1} = x_i^2$ for $i = 1, 2, 3$. Let $\mathscr{H} = \langle x_1, x_2, x_3 \rangle$. Then $\mathscr{H} \triangleleft \mathscr{G}$ and $\mathscr{H}$ is a Sylow 7-subgroup of $\mathscr{G}$. Since $\mathscr{H}$ is contained in the commutator subgroup of $\mathscr{G}$, $\mathscr{G}$ has no homomorphic images of order 7. $\mathscr{G}$ is $\mathbb{Q}(t)$-admissible by Theorem 4 and is not $\mathbb{Q}((t))$-admissible by Lemma 6. □

As mentioned earlier, any homomorphic image of a $\mathbb{Q}$-admissible group is necessarily also $\mathbb{Q}$-admissible [Sc, Corollary 2.3]. We next show that the analogous result for $\mathbb{Q}((t))$-admissibility is false.

Example. Let p be an odd prime and suppose that $\mathscr{P}_0$ is the non-abelian group of order p^3 and exponent p. Let $\mathscr{H} = \mathscr{P}_0 \times (\mathbb{Z}/p\mathbb{Z})$ and let $\mathscr{G} = \mathscr{P}_0 \times (\mathbb{Z}/p^3\mathbb{Z})$. Then $\mathscr{G}$ is $\mathbb{Q}((t))$-admissible, $\mathscr{H}$ is a homomorphic image of $\mathscr{G}$, and $\mathscr{H}$ is not $\mathbb{Q}((t))$-admissible.

Proof. $\mathscr{G}$ is $\mathbb{Q}((t))$-admissible by Corollary 5. Since $\mathscr{H}$ is clearly a homomorphic image of $\mathscr{G}$, we need only show that $\mathscr{H}$ is not $\mathbb{Q}((t))$-admissible. Suppose it is. Then there exists a $\mathbb{Q}((t))$-division algebra D of index p^4 possessing a maximal subfield $\widehat{E}$ with $\widehat{E}/\mathbb{Q}((t))$ $\mathscr{H}$-Galois. By Lemma 2, $\widehat{E} = E((t))$ for some $\mathscr{H}$-Galois extension E of $\mathbb{Q}$. By Proposition 1(1), there exists a $\mathbb{Q}$-division algebra D_0 of p-power Schur index, a cyclic p-extension L of $\mathbb{Q}$, and a generator σ for $\mathrm{Gal}(L/\mathbb{Q})$ such that $[D] = [(D_0 \otimes_{\mathbb{Q}} \mathbb{Q}((t)))] + [(L((t))/\mathbb{Q}((t)), \sigma, t)]$. By Proposition 1(3), $p^4 = \mathrm{ind}(D) = \mathrm{ind}(D_0 \otimes_{\mathbb{Q}} L) \cdot [L : \mathbb{Q}]$. Since $E((t))$ splits D, E splits D_0 and $E \supset L$ by Proposition 1(2). Since $\mathscr{H}$ has exponent p and $\mathrm{Gal}(L/\mathbb{Q})$ is a homomorphic image of $\mathscr{H}$, $[L : \mathbb{Q}] = 1$ or p. Thus $\mathrm{ind}(D_0) \ge \mathrm{ind}(D_0 \otimes_{\mathbb{Q}} L) \ge p^3$. It follows that there exists a rational prime $q \neq p$ such that the local index of D_0 at q is $\ge p^3$. Since E splits D_0, the local degree of E over $\mathbb{Q}$ at q must be $\ge p^3$. Since $q \neq p$, the local extensions of E over $\mathbb{Q}$ at q are tamely ramified and so the local Galois groups are meta-cyclic [W, Theorem 3-5-3 and Proposition 3-6-4]. Since the local Galois groups are subgroups of $\mathscr{H}$, it follows that $\mathscr{H}$ must possess metacyclic subgroups of order $\geq p^3$. Since $\mathscr{H}$ has exponent p, this is impossible and so $\mathscr{H}$ is not $\mathbb{Q}((t))$-admissible. □

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