PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

ANTIPODAL COINCIDENCE FOR MAPS OF SPHERES INTO COMPLEXES

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(Communicated by Thomas Goodwillie)

ABSTRACT. This paper gives a partial answer to the question of whether there exists a Borsuk-Ulam type theorem for maps of S^n into lower-dimensional spaces, which are not necessarily manifolds. It is shown that for each k and $n \leq 2k-1$, there exists a map f of S^n into a contractible k-dimensional complex Y such that $fx \neq f(-x)$, for all $x \in S^n$. In particular, there exists a map of S^3 into a 2-dimensional complex Y without an antipodal coincidence. This answers a question raised by Conner and Floyd in 1964. The complex Y provides also an example of a countractible k-dimensional complex whose deleted product has the Yang-index equal to $2k-1$.

1. INTRODUCTION AND NOTATION

We will say that a map f from S^n to a space Y has an *antipodal coincidence* if there exists a point $x \in S^n$ such that $fx = f(-x)$. The classical Borsuk-Ulam theorem says that every map $f\colon S^n \to \mathbf{R}^k$ has an antipodal coincidence if $k \leq n$. Conner and Floyd ([1], p. 85) proved that if $k < n$, then every map $f\colon S^n \to M^k$ of S^n into a k-dimensional differentiable manifold M^k has an antipodal coincidence. They also asked a specific question ([1], 89) of whether there is a map of S^3 into a 2-dimensional complex without an antipodal coincidence. We will construct such a map of S^3 into a 2-dimensional contractible complex Y. The construction is extremely simple and geometric: in our example, Y is a subcomplex of the barycentric subdivision of a standard simplex. We will also show that 3 is the lowest dimension of the sphere for which such an example can be constructed.

If A and B are spaces, we will denote by $A * B$ the join of A and B. It consists of segments joining the points of A with the points of B, the segments being mutually disjoint except at the endpoints. The join of A with the empty set is A itself. There is a standard (deformation) retraction of $(A * B) - A$ to B along the segments of the join.

If s is a simplex and t is a face of s, we will denote by $c(t)$ the face of s which is *complementary* to t; i.e., s is the join of t and $c(t)$; and

Received by the editors June 8, 1993 and, in revised form, September 28, 1993 and October 12, 1993.

1991 *Mathematics Subject Classification*. Primary 55M20; Secondary 55M35.

Work by the first author was supported, in part, by the Kosciuszko Foundation.

$\dim c(t) = \dim s - \dim t - 1$. The barycenter of a simplex s will be denoted by $b(s)$. By the *carrier* of a point x in a complex K we mean, as usual, the simplex of K containing x in the interior.

Let Δ be a standard $(n+1)$-dimensional simplex and let Δ' denote its barycentric subdivision. We can think of Δ as a subset of $\mathbf{R}^{n+2}$ embedded in the standard way, so that the coordinates in $\mathbf{R}^{n+2}$ are the barycentric coordinates of points of Δ. We will denote by $\alpha\colon \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ the symmetry of $\mathbf{R}^{n+2}$ about the diagonal of $\mathbf{R}^{n+2}$, i.e., the line where all the coordinates are equal to each other. Then α corresponds to the antipodal map of the $(n+1)$-plane P in which the simplex Δ lies. Let o be the barycenter of Δ. Two points of $\mathbf{R}^{n+2}$ are said to be *antipodal* (to each other) if they lie on opposite sides of the center o. The simplex Δ is not invariant under the central symmetry α, but the symmetry induces a simplicial map $\beta\colon \Delta' \to \Delta'$ of the barycentric subdivision of Δ. In fact, if $b = b(s)$ is a vertex of Δ' and b is the barycenter of a face s of Δ, then $\beta(b)$ is the barycenter of the face complementary to b: $\beta(b) = b(c(s))$.

The vertices of the barycentric subdivision Δ' are partially ordered, as usual, by their "ranks": *the rank* of a vertex b of Δ' is the dimension of the face of Δ of which b is the barycenter. Note the following facts:

Lemma 1. (1) *The involution $\beta\colon \Delta' \to \Delta'$ reverses the ranks of the vertices: if b is a vertex of Δ' of rank q, then the rank of $\beta(b)$ is $n-q$ (in the case $q = n+1$ it is understood that the center o is the barycenter of both Δ and the empty simplex).*

(2) *$\beta\colon \Delta' \to \Delta'$ is simplicially free outside the center o in the sense that for each simplex s of Δ', the intersection of s with $\beta(s)$ consists at most of the center o.*

(3) *If x and y is an antipodal pair in Δ and if u and v are the carriers of x and y in Δ', respectively, then $v = \beta(u)$.* □

The construction of our example rests on the fact that a simplex is so much "antisymmetric" with respect to the antipodal map.

Lemma 2. *If s is a q-simplex of Δ' which is a part of the q-skeleton Δ^q of Δ, then the intersection $(\beta(s)) \cap \Delta^q$ is a face of $\beta(s)$ of dimension $2q-n$.*

Proof. Since s is a simplex of Δ' in Δ^q, the ranks of its vertices form a sequence $(0, \dots, q)$. Thus by Lemma 1, the ranks of $\beta(s)$ form a sequence $(n-q, \dots, n)$. The face of $\beta(s)$ which is in Δ^q is made up of the vertices whose ranks are not more than q. In the sequence $(n-q, \dots, n)$ there are exactly $2q-n+1$ integers not greater than q. □

Corollary. *If $2q+1 \le n$, then $\beta(\Delta^q) \cap \Delta^q = \varnothing$.*

2. Construction of the complex Y

Throughout Sections 2 and 3, let q be an integer and $k = n-q$.

Definition. Every simplex s of the barycentric subdivision Δ' is the join of a unique pair u, v of simplices of Δ', where u lies in the q-skeleton Δ^q of Δ and v is disjoint from the q-skeleton. The simplices v so obtained form a k-dimensional subcomplex Y^k of Δ'. We will call it the *k-dimensional spine*, denoted Y^k, of Δ (briefly, the *k-spine*) "dual" to the q-skeleton Δ^q. We can also define the spine inductively by saying that the k-spine of a simplex is the union of the cones over the $(k-1)$-spines of the boundary simplices.

Let $v_0, \dots, v_{n+1}$ be the vertices of Δ. Given a set of $q+1$ indices $i_0, \dots, i_q$ out of $(0, \dots, n+1)$, we will denote by $b(i_0, \dots, i_q)$ the center of the face spanned by $v_{i_0}, \dots, v_{i_q}$. The k-spine Y^k is disjoint with the q-skeleton of Δ and intersects each $(q+1)$-face of Δ spanned by $v_{i_0}, \dots, v_{i_{q+1}}$ in its center $b(i_0, \dots, i_{q+1})$. These centers are "the extremities" of the spine Y^k. There are $\binom{n+2}{q+2}$ of them.

Let C_k be the convex hull of the k-spine Y^k. Then C_k is also the convex hull of the set of its extremities $\{b(i_0, \dots, i_{q+1})\}$. It is a convex $(n+1)$-dimensional polyhedron with vertices $\{b(i_0, \dots, i_{q+1})\}$ contained in $\Delta - \Delta^q$. The boundary of C_k, which we denote by Σ_k, is a simplicial n-sphere.

Note that C_k is the intersection of the $(n+2)$ half-spaces given by $(q+2)x_i \leq 1$, $i = 0, \dots, n+1$, with the simplex Δ. The part of C_k lying on an m-face of C_k is just $C_{k+m-n-1}$.

The special case of $n = 3$, $k = 2$, $q = 1$. In this case (which is actually the first interesting one), we can see that Y^2 is a 2-dimensional complex whose intersection with each 3-dimensional simplex s on the boundary of Δ consists of four intervals going from the center of s to its 2-dimensional faces. The complex Y^2 itself is the cone over the graph made up by these four-legged objects. The convex hull C_2 of Y^2 is a 4-dimensional polyhedron with 10 vertices. It cannot be a regular polyhedron since there are no regular 4-dimensional polyhedra with 10 vertices. In fact, the boundary Σ_2 of C_2 has five tetrahedra (one in every 3-simplex of Σ_2), five octahedra (one "opposite" each vertex of Δ), thirty triangles, thirty edges and ten vertices (the Euler number being zero, as it should be).

3. The map of S^n into Y^k

Recall that every simplex s of Δ' is the join of a unique simplex u of the barycentric subdivision of the q-skeleton of Δ^q of Δ with a unique simplex v of Y^k. Thus there exists a standard deformation retraction of $\Delta - \Delta^q$ to Y^k along the lines of the join in each simplex of Δ' (with its face in Δ^q deleted). Let $f\colon \Sigma_k \cong S^n \to Y^k$ be the restriction of this map to Σ_k. The antipodal symmetry α induces a free involution on $\Sigma_k \cong S^n$ which will also be called antipodal.

Lemma 3. *If x and y is a pair of antipodal points of Σ_k such that $fx = fy$, then $fx = fy$ must be the center o.*

Proof. Let $z = fx = fy$. Let t be the carrier of z in Y^k and let u and v be the carriers of x and y, respectively, in Δ'. By the construction of f, the simplex t is a face of both u and v. Since x and y is an antipodal pair, Lemma 1 implies that the intersection $u \cap v$ is just the center o. Thus $z = o$. □

Proposition. *If $n \leq 2k-1$, then the map $f\colon S^n \to Y^k$ has no antipodal coincidence: if x and y is an antipodal pair on S^n, then $fx \neq fy$.*

Proof. By Lemma 3, f projects the points x and y to the center o from an antipodal pair of points lying in the $(n-k)$-skeleton Δ^{n-k} of Δ. However, if

$n \leq 2k - 1$, then $2(n-k) - 1 \leq n$ and by the corollary, Δ^{n-k} contains no antipodal pair. $\square$

4. Concluding remarks

It is not hard to see that if $n > 2k$, then every map $f: S^n \to Y$ of a finite k-dimensional complex Y into S^n has an antipodal coincidence. If, in addition, $H_k(Y; \mathbf{Z}_2) = 0$, then the inequality $n > 2k - 1$ is already sufficient. For, if $f: S^n \to Y$ is a map without an antipodal coincidence, then f defines a $\mathbf{Z}_2$-equivariant map from S^n to the deleted product Y^* of Y. By an argument similar to that used in [2] one can show that $H^{2k}(Y^*; \mathbf{Z}_2) = 0$. Thus the Yang $\mathbf{Z}_2$-index, $\operatorname{Ind}^{\mathbf{Z}_2} Y^*$, of Y^* (see [1] and [3]) is less than $2k \leq n$ and hence an equivariant map from S^n to Y^* cannnot exist.

Thus we have the following result.

Theorem. *For each k and $n \leq 2k - 1$, there exists a map f of S^n into a contractible k-dimensional complex Y without an antipodal coincidence. In particular, there exists such a map of S^3 into a contractible 2-dimensional complex. Moreover, $n = 3$ is the lowest integer for which there is a map of S^n into a complex of dimension less than n without an antipodal coincidence.* $\square$

In particular, the deleted produce Y^* of the contractible complex Y constructed in Section 2 has the Yang $\mathbf{Z}_2$-index equal to $2k - 1$. We do not know whether there exist a k-dimensional complex whose deleted product has the Yang $\mathbf{Z}_2$-index equal to $2k$.

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