

## NORMAL DERIVATIONS IN NORM IDEALS

FUAD KITTANEH

(Communicated by Palle E. T. Jorgensen)

**ABSTRACT.** We establish the orthogonality of the range and the kernel of a normal derivation with respect to the unitarily invariant norms associated with norm ideals of operators. Related orthogonality results for certain nonnormal derivations are also given.

### 1. INTRODUCTION

Let  $B(H)$  denote the algebra of all bounded linear operators on an infinite-dimensional complex separable Hilbert space  $H$ . For operators  $A, B$  in  $B(H)$ , the generalized derivation  $\delta_{A,B}$  as an operator on  $B(H)$  is defined by

$$(1) \quad \delta_{A,B}(X) = AX - XB \quad \text{for all } X \in B(H).$$

When  $A = B$ , we simply write  $\delta_A$  for  $\delta_{A,A}$ . If  $N$  is a normal operator in  $B(H)$ , then  $\delta_N$  is said to be a normal derivation.

In his investigation of normal derivations, Anderson [1, Theorem 1.7] proved that if  $N$  and  $S$  are operators in  $B(H)$  such that  $N$  is normal and  $NS = SN$ , then for all  $X \in B(H)$

$$(2) \quad \|\delta_N(X) + S\| \geq \|S\|,$$

where  $\|\cdot\|$  is the usual operator norm. Thus in the sense of [1, Definition 1.2], inequality (2) says that the range of  $\delta_N$  is orthogonal to the kernel of  $\delta_N$ , which is just the commutant  $\{N\}'$  of  $N$ .

It has been shown in [11, Theorem 1] that if  $N$  and  $S$  are operators in  $B(H)$  such that  $N$  is normal,  $S$  is a Hilbert-Schmidt operator, and  $S \in \{N\}'$ , then for all  $X \in B(H)$

$$(3) \quad \|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2,$$

where  $\|\cdot\|_2$  is the Hilbert-Schmidt norm. Thus in the usual Hilbert space sense, the Hilbert-Schmidt operators in the range of  $\delta_N$  are orthogonal to those in the kernel of  $\delta_N$ .

---

Received by the editors July 22, 1993 and, in revised form, September 28, 1993.

1991 *Mathematics Subject Classification.* Primary 47A30, 47B10, 47B15, 47B20, 47B47; Secondary 46B20.

*Key words and phrases.* Normal derivation, nonnormal derivation, norm ideal, unitarily invariant norm, orthogonality results for derivations.

It has also been shown recently in [12, Theorem 3.2] that if  $N$  and  $S$  are operators in  $B(H)$  such that  $N$  is normal and  $S$  belongs to some Schatten  $p$ -class  $C_p$  with  $1 \leq p \leq \infty$  and  $S \in \{N\}'$ , then for all  $X \in B(H)$

$$(4) \quad \|\delta_N(X) + S\|_p \geq \|S\|_p.$$

The usual operator norm, the Hilbert-Schmidt norm, and the Schatten  $p$ -norms are only examples of a large family of unitarily invariant (or symmetric) norms on  $B(H)$ .

The purpose of this paper is to investigate the orthogonality of the range and the kernel of a normal derivation with respect to the wider class of unitarily invariant norms on  $B(H)$ . Derivations induced by certain nonnormal operators will also be discussed.

In §2 we will use a completely different analysis to extend (4) to all unitarily invariant norms defined on norm ideals of compact operators in  $B(H)$ . Extensions of this result to certain nonnormal operators will be the main theme of §3, in which we will treat derivations of the form  $\delta_{A,B}$ , where  $A$  is a dominant operator and  $B^*$  is  $M$ -hyponormal. Moreover we will discuss the validity of (2) for various classes of derivations at the expense of requiring that  $S$  is normal. A relevant example will also be presented.

Recall that each unitarily invariant norm  $|||\cdot|||$  is defined on a natural subclass  $J_{|||\cdot|||}$  of  $B(H)$  called the norm ideal associated with the norm  $|||\cdot|||$  and satisfies the invariance property  $|||UAV||| = |||A|||$  for all  $A \in J_{|||\cdot|||}$  and for all unitary operators  $U, V \in B(H)$ . While the usual operator norm  $\|\cdot\|$  is defined on all of  $B(H)$ , the other unitarily invariant norms are defined on norm ideals contained in the ideal of compact operators in  $B(H)$ . Given any compact operator  $A \in B(H)$ , denote by  $s_1(A) \geq s_2(A) \geq \cdots$  the singular values of  $A$ , i.e., the eigenvalues of  $|A| = (A^*A)^{1/2}$ . There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if  $|||\cdot|||$  is a unitarily invariant norm, then there is a unique symmetric gauge function  $\Phi$  such that

$$(5) \quad |||A||| = \Phi(\{s_j(A)\})$$

for all  $A \in J_{|||\cdot|||}$ .

For  $1 \leq p \leq \infty$ , define

$$(6) \quad \|A\|_p = \left( \sum_j s_j^p(A) \right)^{1/p},$$

where, by convention,  $\|A\|_\infty = s_1(A)$  is the usual operator norm of the compact operator  $A$ . These unitarily invariant norms are the well-known Schatten  $p$ -norms associated with the Schatten  $p$ -classes  $C_p$ ,  $1 \leq p \leq \infty$ . Hence  $C_1$ ,  $C_2$ , and  $C_\infty$  are the trace class, the Hilbert-Schmidt class, and the class of compact operators, respectively. For good accounts on the theory of norm ideals and their associated unitarily invariant norms, the reader is referred to [9], [13], or [14] (see also [2] and references therein).

## 2. NORMAL DERIVATIONS

In this section we present our main result of this paper. This result asserts that if  $N$  is a normal operator in  $B(H)$ , then with respect to any unitarily

invariant norm  $||| \cdot |||$ ,  $\text{ran } \delta_N \cap J_{||| \cdot |||}$  is orthogonal to  $\ker \delta_N \cap J_{||| \cdot |||}$ , where  $\text{ran } \delta_N$  and  $\ker \delta_N$  are the range and the kernel of  $\delta_N$ , respectively.

To accomplish our goal we need two lemmas.

**Lemma 1.** *Let  $N \in B(H)$  be diagonal (normal with pure point spectrum),  $S \in \{N\}'$ , and  $X \in B(H)$ . If  $\delta_N(X) + S \in J_{||| \cdot |||}$ , then  $S \in J_{||| \cdot |||}$  and*

$$(7) \quad |||\delta_N(X) + S||| \geq |||S|||.$$

*Proof.* Let  $N$  have the distinct eigenvalues  $\lambda_1, \lambda_2, \dots$ . Then, with respect to the decomposition  $H = \bigoplus_{j=1}^{\infty} \ker(N - \lambda_j)$ ,  $N$  has the operator matrix representation

$$N = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \end{bmatrix}.$$

Let  $[S_{ij}]$  and  $[X_{ij}]$  be the matrix representations of  $S$  and  $X$  with respect to the above decomposition of  $H$ . Then  $NX - XN = [(\lambda_i - \lambda_j)X_{ij}]$ , and in view of the assumption  $S \in \{N\}'$  we have  $S_{ij} = 0$  for  $i \neq j$ . Therefore,

$$NX - XN + S = \begin{bmatrix} S_{11} & & * \\ & S_{22} & \\ & & \ddots \\ * & & \end{bmatrix}.$$

Since  $\delta_N(X) + S \in J_{||| \cdot |||}$  and since the norm of an operator matrix always dominates the norm of its diagonal part (see [9, p. 82]), it follows that

$$S \in J_{||| \cdot |||} \text{ and } |||\delta_N(X) + S||| \geq |||S|||.$$

**Lemma 2.** *Let  $N \in B(H)$  be normal, and set  $H_1 = \bigvee_{\lambda \in \sigma_p(N)} \ker(N - \lambda)$ . If  $S \in \{N\}'$  and there is an  $X \in B(H)$  such that  $\delta_N(X) + S \in C_{\infty}$ , then  $H_1$  reduces  $S$  and  $S|_{H_1^{\perp}} = 0$ .*

*Proof.* Since  $N$  is normal,  $H_1$  reduces  $N$  and  $N|_{H_1}$  is a diagonal operator. By Fuglede's theorem (see [10, p. 104])  $S^* \in \{N\}'$ , so  $H_1$  also reduces  $S$ . Let

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

on  $H = H_1 \oplus H_2$ , where  $H_2 = H_1^{\perp}$ . The assumption  $\delta_N(X) + S \in C_{\infty}$  implies  $\delta_{N_2}(X_{22}) + S_2 \in C_{\infty}$ . Anderson's result (2) (applied to the Calkin algebra  $B(H_2)/C_{\infty}$ ) insures that  $S_2 \in C_{\infty}$ . Since the normal operator  $N_2$  has no eigenvalues and since the compact selfadjoint operator  $S_2^*S_2$  belongs to  $\{N_2\}'$ , it follows that  $S_2^*S_2 = 0$ . Hence  $S_2 = 0$ , as desired.

Now we are in a position to prove the main result of this paper.

**Theorem 1.** *Let  $N \in B(H)$  be normal,  $S \in \{N\}'$ , and  $X \in B(H)$ . If  $\delta_N(X) + S \in J_{||| \cdot |||}$ , then  $S \in J_{||| \cdot |||}$  and*

$$(8) \quad |||\delta_N(X) + S||| \geq |||S|||.$$

*Proof.* Since  $\delta_N(X) + S \in J_{||| \cdot |||} \subseteq C_{\infty}$ , it follows by Lemma 2 that on  $H = H_1 \oplus H_1^{\perp}$ ,

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where  $H_1 = \bigvee_{\lambda \in \mathbb{C}} \ker(N - \lambda)$ . If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

on  $H = H_1 \oplus H_1^\perp$ , then

$$\delta_N(X) + S = \begin{bmatrix} \delta_{N_1}(X_{11}) + S_1 & * \\ * & * \end{bmatrix}.$$

Since  $\delta_N(X) + S \in J_{||\cdot||}$ , it follows that  $\delta_{N_1}(X_{11}) + S_1 \in J_{||\cdot||}$ . But  $N_1$  is diagonal and  $S_1 \in \{N_1\}'$ . Thus, by Lemma 1,  $S_1 \in J_{||\cdot||}$  and  $||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1||$ . Consequently,  $S \in J_{||\cdot||}$  and  $||\delta_N(X) + S|| \geq ||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1|| = ||S||$ .

At the end of this section we use a familiar device of considering  $2 \times 2$  operator matrices to extend Theorem 1 to generalized normal derivations.

**Corollary 1.** Let  $N, M, S \in B(H)$  such that  $N$  and  $M$  are normal and  $NS = SM$ . If  $X \in B(H)$  such that  $\delta_{N,M}(X) + S \in J_{||\cdot||}$ , then  $S \in J_{||\cdot||}$  and

$$(9) \quad ||\delta_{N,M}(X) + S|| \geq ||S||.$$

*Proof.* On  $H \oplus H$  consider the operators  $L, T$ , and  $Y$  defined as

$$L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \quad T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then  $L$  is normal,  $T \in \{L\}'$ , and

$$\delta_L(Y) + T = \begin{bmatrix} 0 & \delta_{A,B}(X) & + & S \\ 0 & 0 & & \end{bmatrix}.$$

Thus by Theorem 1 applied to the operators  $L, T$ , and  $Y$  we have  $T \in J_{||\cdot||}$  and  $||\delta_L(Y) + T|| \geq ||T||$ . Therefore  $S \in J_{||\cdot||}$  and  $||\delta_{A,B}(X) + S|| \geq ||S||$ , as desired.

### 3. NONNORMAL DERIVATIONS

Extensions of (3) to certain subnormal operators have been given in [11, Theorems 2 and 3]. In the same vein we devote this section to the extension of the results in §2 to classes of operators larger than that of normal operators.

Recall that an operator  $A \in B(H)$  is called dominant (see [15]) if

$$(10) \quad \text{ran}(A - \lambda) \subseteq \text{ran}(A - \lambda)^* \quad \text{for all } \lambda \in \mathbb{C}.$$

In view of [6],  $A$  is dominant if and only if for each  $\lambda \in \mathbb{C}$  there exists a constant  $M_\lambda$  such that

$$(11) \quad \|(A - \lambda)^* f\| \leq M_\lambda \|(A - \lambda)f\| \quad \text{for all } f \in H.$$

If there is a constant  $M$  such that  $M_\lambda \leq M$  for all  $\lambda \in \mathbb{C}$ , then  $A$  is called  $M$ -hyponormal. If  $M = 1$ , then  $A$  is hyponormal.

Our promised generalization of (9) can be stated as follows.

**Theorem 2.** Let  $A, B, S \in B(H)$  such that  $A$  is dominant,  $B^*$  is  $M$ -hypo-normal, and  $AS = SB$ . If  $X \in B(H)$  such that  $\delta_{A,B}(X) + S \in J_{||| \cdot |||}$ , then  $S \in J_{||| \cdot |||}$  and

$$(12) \quad |||\delta_{A,B}(X) + S||| \geq |||S|||.$$

*Proof.* Since the pair  $(A, B)$  satisfies the Fuglede-Putnam property, it follows (see [16] or [19]) that  $\overline{\text{ran } S}$  (the closure of  $\text{ran } S$ ) reduces  $A$ ,  $\ker^\perp S$  (the orthogonal complement of  $\ker S$ ) reduces  $B$ , and  $A|_{\overline{\text{ran } S}}$  and  $B|_{\ker^\perp S}$  are unitarily equivalent normal operators. Then, with respect to the orthogonal decompositions  $H = \overline{\text{ran } S} \oplus (\overline{\text{ran } S})^\perp$  and  $H = \ker^\perp S \oplus \ker S$ ,  $A$  and  $B$  can be respectively represented as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Now assume that the operators  $S, X: \ker^\perp S \oplus \ker S \rightarrow \overline{\text{ran } S} \oplus (\overline{\text{ran } S})^\perp$  have the matrix representations

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Then  $A_1$  and  $B_1$  are normal, and  $A_1 S_1 = S_1 B_1$ .

Applying Corollary 1 to the operators  $A_1, B_1, S_1$ , and  $X_1$  we see that  $S_1 \in J_{||| \cdot |||}$ . Hence  $S \in J_{||| \cdot |||}$  and

$$\begin{aligned} |||\delta_{A,B}(X) + S||| &= \left\| \begin{bmatrix} \delta_{A_1, B_1}(X_1) + S_1 & * \\ * & * \end{bmatrix} \right\| \\ &\geq |||\delta_{A_1, B_1}(X_1) + S_1||| \geq |||S_1||| = |||S|||, \end{aligned}$$

which completes the proof of the theorem.

The usual operator norm version of (12) has been obtained by Elalami [7, Theorem 4.1] using a different method.

We would like to point out here that in view of [16] Theorem 2 is still valid for any pair of operators  $(A, B)$  which satisfies the Fuglede-Putnam property, that is,  $A^*S = SB^*$  whenever  $AS = SB$ , where  $S \in B(H)$ . For several such pairs, the reader is referred to [4] and references therein.

A closer look at the proof of Theorem 2 (see also [5, Theorem 1]) leads us to show that if  $(A, B)$  satisfies the Fuglede-Putnam property and if  $S \in C_2$  such that  $AS = SB$ , then for all  $X \in B(H)$  we have

$$(13) \quad \|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2.$$

This Hilbert space orthogonality result strengthens (3) and [11, Theorem 3].

It has been shown in [11, Theorem 4] that if  $A \in B(H)$  is a cyclic subnormal operator and if  $S \in C_2 \cap \{A\}'$ , then for all  $X \in B(H)$  we have

$$(14) \quad \|\delta_A(X) + S\|_2^2 = \|\delta_A(X)\|_2^2 + \|S\|_2^2.$$

In the same direction, it should be noted that the proof of Theorem 2 can be modified to insure that if  $A \in B(H)$  is a cyclic subnormal operator and  $S \in J_{||| \cdot |||} \cap \{A\}'$ , then for all  $X \in B(H)$  we have

$$(15) \quad |||\delta_A(X) + S||| \geq |||S|||.$$

To verify (15) we need only show that  $\overline{\text{ran } S}$  reduces  $A$  and  $A|_{\overline{\text{ran } S}}$  is normal, for then we can follow the arguments in the proof of Theorem 2. Since  $S \in \{A\}'$  and  $A$  is a cyclic subnormal operator, it follows by Yoshino's result [18] that  $S$  is also subnormal. This, together with the assumption  $S \in J_{||\cdot||} \subseteq C_\infty$ , implies that  $S$  is in fact normal. Consequently  $S \in \{A, A^*\}'$ , and so  $\overline{\text{ran } S}$  reduces  $A$ . If  $T = AS^*$ , then  $T \in \{A\}'$  and  $T^*T - TT^* = S(A^*A - AA^*)S^* \geq 0$  (because  $A^*A - AA^* \geq 0$ ). Thus  $T$  is a compact hyponormal operator, and hence  $T$  is normal. Now we have  $AA^*S = ASA^* = AT^* = T^*A = SA^*A = A^*AS$ , and so  $A|_{\overline{\text{ran } S}}$  is normal.

In [8] an example is given to show that the cyclicity assumption on  $A$  is necessary for (14) to hold. This gives an affirmative answer to a question raised in [11]. The following example, which will also be used later in the paper, is simpler and shorter than the one given in [8].

**Example.** Let  $U$  be the unilateral shift operator of multiplicity one. On  $H \oplus H$ , let  $A = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$ ,  $S = \begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix}$ , and  $X = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix}$ , where  $P = 1 - UU^*$  and  $Q = PU^*$ . Then  $A$  is a noncyclic subnormal operator,  $S \in \{A\}'$ , and  $\delta_A(X) + S = 0$ , yet  $|||\delta_A(X)||| = |||S||| = 1$  for every unitarily invariant norm  $|||\cdot|||$ .

This example also indicates that Anderson's result (2) cannot be extended to derivations induced by subnormal operators. However, if we require  $S$  to be normal, then in this case (2) works for several classes of operators. The list includes compact operators, dominant operators, quasinilpotent operators with positive real parts, and operators  $A$  for which  $p(A) = 0$  for some quadratic polynomial  $p$  (see [7, 17]).

Another interesting class of operators for which (2) is true when  $S$  is normal is the class of operators  $A$  such that  $A^*A$  and  $A + A^*$  commute. It is well known that this class enjoys the property that  $\|A\| = r(A)$  (the spectral radius of  $A$ ) (see [3]). Hence it is elementary to verify that (see [10, p. 130]) for all  $X \in B(H)$  and all  $\lambda \in \mathbb{C}$  we have

$$(16) \quad \|\delta_A(X) + \lambda\| \geq |\lambda|.$$

Based on (16) and the spectral theorem for normal operators it can be shown that if  $A, S \in B(H)$  such that  $A^*A$  commutes with  $A + A^*$ ,  $S$  is normal, and  $S \in \{A\}'$ , then for all  $X \in B(H)$  we have

$$(17) \quad \|\delta_A(X) + S\| \geq \|S\|.$$

To prove (17) we first assume that  $S$  is a normal operator with finite spectrum. Then we use a continuity argument to establish the general case.

Finally, we remark that the example presented above shows that (2) fails to hold for an arbitrary (not necessarily normal) operator  $S$  in the commutant of  $A$ .

#### ACKNOWLEDGMENT

The author thanks Professor K. Takahashi for his helpful suggestions.

#### REFERENCES

1. J. H. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973), 135–140.
2. R. Bhatia and F. Kittaneh, *Norm inequalities for partitioned operators and an application*, Math. Ann. **287** (1990), 719–726.

3. S. L. Campbell and R. Gellar, *Linear operators for which  $T^*T$  and  $T^* + T$  commute*, Proc. Amer. Math. Soc. **60** (1976), 197–202.
4. B. P. Duggal, *On generalised Putnam-Fuglede theorems*, Monatsh. Math. **107** (1989), 309–332.
5. ———, *A remark on normal derivations of Hilbert-Schmidt type*, Monatsh. Math. **112** (1991), 265–270.
6. R. G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.
7. S. N. Elalami, *Opérateurs complètement finis*, preprint.
8. W. Feng and G. Ji, *A counterexample in the theory of derivations*, Glasgow Math. J. **31** (1989), 161–163.
9. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI, 1969.
10. P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York, 1982.
11. F. Kittaneh, *On normal derivations of Hilbert-Schmidt type*, Glasgow Math. J. **29** (1987), 245–248.
12. P. J. Maher, *Commutator approximants*, Proc. Amer. Math. Soc. **115** (1992), 995–1000.
13. R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.
14. B. Simon, *Trace ideals and their applications*, Cambridge Univ. Press, London and New York, 1979.
15. J. G. Stampfli and B. L. Wadhwani, *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25** (1976), 359–365.
16. K. Takahashi, *On the converse of the Fuglede-Putnam theorem*, Acta Sci. Math. (Szeged) **43** (1981), 123–125.
17. J. P. Williams, *Finite operators*, Proc. Amer. Math. Soc. **26** (1970), 129–136.
18. T. Yoshino, *Subnormal operators with a cyclic vector*, Tôhoku Math. J. **21** (1969), 47–55.
19. T. Yoshino, *Remarks on the generalized Putnam-Fuglede theorem*, Proc. Amer. Math. Soc. **95** (1985), 571–572.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JORDAN, AMMAN, JORDAN