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ON THE MULTIPLE POINTS OF CERTAIN MEROMORPHIC FUNCTIONS

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ABSTRACT. We show that if f is transcendental and meromorphic in the plane and $T(r, f) = o(\log r)^2$, then f has infinitely many critical values. This is sharp. Further, we apply a result of Eremenko to show that if f is meromorphic of finite lower order in the plane and $N(r, 1/ff'') = o(T(r, f'/f))$, then $f(z) = \exp(az+b)$ or $f(z) = (az+b)^{-n}$ with a and b constants and n a positive integer.

1. INTRODUCTION

If g is a function transcendental and meromorphic in the plane, then the term

$$N_1(r, g) = N(r, g) - \overline{N}(r, g) + N(r, 1/g'),$$

in which the counting functions are defined as in [11, Chapter 2], counts the multiple points of g. The following has been proved by Eremenko.

Theorem A [5]. *Let g be transcendental and meromorphic in the plane of finite lower order such that $\delta(\infty, g) = 0$ and $N_1(r, g) = o(T(r, g))$. Then there exist an integer $2\rho \geq 2$ and continuous functions $L_1(r)$ and $L_2(r)$ such that $L_1(ct) = L_1(t)(1+o(1))$ and $L_2(ct) = L_2(t) + o(1)$ as $t \to +\infty$, uniformly for $1 \leq c \leq 2$, and such that*

$$-\log|g'(re^{i\theta})| = \pi r^\rho L_1(\rho)|\cos(\rho(\theta - L_2(r)))| + o(r^\rho L_1(r))$$

as $r \to +\infty$, uniformly in θ, $0 \leq \theta \leq 2\pi$, provided that $re^{i\theta}$ lies outside an exceptional set C_0 of discs $B(z_k, r_k)$ with the property that if R is large, then the sum of the radii r_k of the discs $B(z_k, r_k)$ for which $|z_k| < R$ is $o(R)$. Further, $\sum_{a\in\mathbb{C}} \delta(a, g) = 2$ and $T(r, g) = (1+o(1))r^\rho L_1(r)$.

It follows from Theorem A that a transcendental meromorphic function g of order less than 1 cannot satisfy $N_1(r, g) = o(T(r, g))$ and so must have multiple points (Shea [19] had earlier proved this when g has order less than $1/2$). Infinitely many of these multiple points must be zeros of g', as is shown by the the following result from [6].

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Theorem B. *Suppose that g is transcendental and meromorphic in the plane with $T(r, g) = o(r)$. Then g' has infinitely many zeros.*

Further, if g is transcendental meromorphic with $T(r, g) = o(r^{1/2})$ or transcendental entire with $T(r, g) = o(r)$, then g'/g must have zeros. These assertions are proved in [3] and are shown there to be sharp.

This suggests the question as to whether a growth condition on a transcendental meromorphic function f forces f to have infinitely many critical values, that is, values take by f at multiple points of f. If $T(r, f) = o(r^{1/2})$ and f is transcendental with only finitely many poles, it is easily seen from the discussion in [18, pp. 269–272] that ∞ must be an accumulation point of critical values of f, for otherwise the inverse function f^{-1} would have a logarithmic singularity at ∞ and, if R is large, there would exist a simply-connected unbounded component U of the set $\{z \in \mathbb{C}: |f(z)| > R\}$, with $|f(z)| = R$ on the boundary of U, which contradicts the $\cos \pi\rho$ theorem [11, p. 119]. Corresponding to this remark is the obvious example $\cos(\sqrt{z})$.

The above observation and example also appear in [2], of which the author became aware after the first draft of the present paper was written. Among other results in [2] concerning asymptotic and critical values of meromorphic functions, it is shown (Corollary 3) that if the transcendental meromorphic function f has finite order ρ and only finitely many critical values, then the number of asymptotic values of f is at most 2ρ.

While a transcendental entire function always has ∞ as an asymptotic value, by the classical theorem of Iversen [18], meromorphic functions need not have any asymptotic values at all. Bank and Kaufman [1] (see also [13, Chapter 11]) proved the existence of a function f transcendental and meromorphic in the plane with $T(r, f) = O(\log r)^2$, satisfying the differential equation

$$(z^2 - 4)(f'(z))^2 = 4(f(z) - e_1)(f(z) - e_2)(f(z) - e_3),$$

in which the e_j are distinct complex numbers, and this function f clearly has just 4 critical values. This example is obtained from the Weierstrass doubly periodic function. For smaller growth, we prove the following theorem, the proof of which is based on a combination of representations for the function in annuli with the Riemann-Hurwitz formula.

Theorem 1. *If f is transcendental and meromorphic in the plane with $T(r, f) = o(\log r)^2$, then f has infinitely many critical values.*

Our second result is a fairly straightforward application of Theorem A, coupled with a variant of a method of Mues from [17]. It was proved in [14] that if f is meromorphic in the plane and f and f'' have no zeros, then $f(z) = \exp(az + b)$ or $f(z) = (az + b)^{-n}$ with a and b constants and n a positive integer. This proved a conjecture of Hayman [10, 12], the case where f has finite order having been settled by Mues in [17]. The same conclusion holds if f is meromorphic in the plane and $N(r, 1/ff^{(k)}) = o(T(r, f'/f))$ for some $k \geq 3$ [8, Theorem 2; see also 7, 9]. We prove here the following result.

Theorem 2. *Suppose that f is meromorphic of finite lower order in the plane and that a_1 and a_0 are rational functions such that the differential equation*

$$y'' + a_1 y' + a_0 y = 0 \tag{1.1}$$

has linearly independent rational solutions f_1 *and* f_2. *If* $F(z) = f''(z) + a_1(z)f'(z) + a_0(z)f(z)$ *and* $N(r, 1/fF) = o(T(r, f'/f))$, *then* f'/f *is rational and* f *and* F *have no zeros.*

Corollary. *If* f *is meromorphic of finite lower order in the plane and*

$$N(r, 1/ff'') = o(T(r, f'/f)),$$

then $f(z) = \exp(az+b)$ *or* $f(z) = (az+b)^{-n}$ *with* a *and* b *constants and* n *a positive integer.*

The corollary follows at once from Theorem 2 using [11, p. 76]. Note that the assumption that (1.1) has a rational fundamental solution set implies that $a_j(z) = O(|z|^{j-2})$ as $z \to \infty$ and that larger coefficients cannot be allowed in general, as the example $g(z) = \sec(\sqrt{z})$, $G(z) = g''(z) + (1/2z)g'(z) + (1/4z)g(z) = g^3(z)/2z$, shows. In [14] and [15], the author determined all functions f meromorphic in the plane such that f and $f'' + a_1f' + a_0f$ have only finitely many zeros, where a_1 and a_0 are rational. It seems possible that the conclusion of Theorem 2 would be true without any assumption on the growth of f and with a_1 and a_0 any rational functions satisfying $a_j(z) = O(|z|^{j-2})$ (in which case (1.1) might not have solutions meromorphic in a neighbourhood of infinity), but the present proof, which consists of applying Theorem A to

$$H(z) = \frac{f_1'(z) - (f'(z)/f(z))f_1(z)}{f_2'(z) - (f'(z)/f(z))f_2(z)},$$

requires H to be meromorphic in the plane of finite lower order.

2. Preliminaries

A key role in the proof of Theorem 1 is played by the *Riemann-Hurwitz formula* (see [20, Chapter 1]): Suppose that D and G are bounded domains of connectivity m and n respectively and that $f\colon D \to G$ is an analytic function with the property that, for any sequence (z_k) in D, z_k tends to the boundary ∂D as $k \to \infty$ (in the sense that if K is a compact subset of D, then $z_k \in D\backslash K$ for all large enough k) if and only if $f(z_k)$ tends to ∂G. Then there exists a positive integer p such that all values w belonging to G are taken p times in D, counting multiplicities, and such that $m-2 = p(n-2)+r$, where r is the number of critical points of f in D, that is, the number of zeros of f' in D, counting multiplicities.

Suppose now that f is a function meromorphic in the plane with only finitely many critical values. If R and S are large, any bounded component of the set $\{z \in \mathbb{C}\colon R < |f(z)| < S\}$ must be doubly-connected, while any bounded component of the set $\{z \in \mathbb{C}\colon |f(z)| > S\}$ contains one (possibly multiple) pole of f and is simply-connected.

Lemma 1. *Let* $n(t)$ *be nondecreasing, integer valued, and continuous from the right such that* $n(1) = 0$ *and* $n(t) = o(\log t)$ *as* $t \to +\infty$. *Set*

$$h(r) = \int_1^r t\,dn(t).$$

If δ *is a positive constant, then the set* $E(\delta) = \{r \geq 1\colon h(r) \geq \delta r\}$ *has logarithmic density* 0.

Proof. Let $\chi(t)$ be the characteristic function of $E(\delta)$, so that $\chi(t) = 1$ if $t \geq 1$ and $t \in E(\delta)$ and $\chi(t) = 0$ otherwise. Then

$$\int_1^r \chi(t)/t\,dt \leq (1/\delta)\int_1^r h(t)/t^2\,dt = (1/\delta)\int_1^r 1/t\,dh(t) - h(r)/\delta r$$
$$\leq (1/\delta)\int_1^r dn(t) = o(\log r),$$

which is precisely what is asserted in the lemma.

The next lemma is part of a special case of the lemma from [16].

Lemma A. *Let $m(t)$ be nondecreasing, integer-valued and continuous from the right, with $m(1) = 0$ and $m(t) = O(t)$ as $t \to +\infty$. Let $M > 3$ be a constant. Then there exists a set E_M of lower logarithmic density at least $1 - 3/M$; that is,*

$$\int_1^r \chi(t)/t\,dt > (1 - 3/M - o(1))\log r \quad \text{as } r \to +\infty,$$

with $\chi(t)$ the characteristic function of E_M, such that, for $r \in E_M$ and $t \geq r$, we have $m(t)/m(r) \leq (t/r)^{4M}$.

Lemma 2. *Let f be transcendental and meromorphic in the plane with $T(r, f) = o(\log r)^2$. Then there exist sequences R_ν and S_ν tending to $+\infty$, nonzero constants C_ν and D_ν, and integers m_ν and n_ν such that for*

$$R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu \tag{2.1}$$

we have

$$f(z) = C_\nu z^{m_\nu}(1 + o(1)) \tag{2.2}$$

and

$$f'(z) = D_\nu z^{n_\nu}(1 + o(1)). \tag{2.3}$$

Proof. We write $f(z) = U(z)F(z)$ and $f'(z) = V(z)G(z)$ where U and V are rational functions and F and G satisfy $F(0) = G(0) = 1$ and have no zeros or poles in $|z| \leq 1$. We choose a small positive δ and apply Lemma 1 with $n(t) = n(t, 1/F) + n(t, F) + n(t, 1/G) + n(t, G) = O(T(t^2, f)/\log t) = o(\log t)$. Further, we apply Lemma 2 with $M = 100$ and $m(t) = 2^{n(t)}$. This gives arbitrarily large r such that

$$h(r) = \int_1^r t\,dn(t) < \delta r \tag{2.4}$$

and, for $t \geq r$,

$$n(t) - n(r) \leq M_1 \log(t/r), \tag{2.5}$$

where $M_1 = 400/\log 2$. Since $n(t)$ is integer-valued, (2.4) implies that f and f' have no zeros or poles in $\delta r \leq |z| \leq r$. Suppose that

$$\delta^{3/4} r \leq |z| \leq \delta^{1/4} r. \tag{2.6}$$

We write $F(z) = F_1(z)/f_2(z)$ with the f_j entire and $f_1(z) = \prod_{j=1}^{\infty}(1 - z/a_j)$, where the a_j are the zeros of f in $1 < |z| < \infty$, repeated according to multiplicity. For z as in (2.6) we have

$$f_1(z) = z^{n(r, 1/F)} \textstyle\prod_1(-1/a_j)\prod_1(1 - a_j/z)\prod_2(1 - z/a_j), \tag{2.7}$$

in which $\prod_1$ denotes the product over all a_j with $|a_j| < r$ and $\prod_2$ denotes the product over the remaining a_j. With $\sum_1$ defined analogously to $\prod_1$, we have, using (2.4),

$$
\begin{aligned}
|\textstyle\prod_1(1 - a_j/z) - 1| &\le \exp(\textstyle\sum_1 |a_j/z|) - 1 \\
&\le \exp(h(r)/|z|) - 1 \le \exp(\delta r/|z|) - 1 \\
&\le \exp(\delta^{1/4}) - 1 .
\end{aligned} \tag{2.8}
$$

Further, (2.5) gives $n(t, 1/f) - n(r, 1/f) \le M_1(\log(t/r))$ for $t \ge r$, and we have

$$
\begin{aligned}
|\textstyle\prod_2(1 - z/a_j) - 1| &\le \exp\left(|z| \int_r^\infty \frac{1}{t}\, dn(t, 1/f)\right) - 1 \\
&= \exp\left(|z| \int_r^\infty [n(t, 1/f) - n(r, 1/f)]\, dt/t^2\right) - 1 \\
&\le \exp\left(|z| M_1 \int_r^\infty \log(t/r)\, dt/t^2\right) - 1 \\
&= \exp(|z|M_1/r) - 1 \le \exp(M_1\delta^{1/4}) - 1 .
\end{aligned} \tag{2.9}
$$

Now if $\varepsilon > 0$ is given, we need only choose δ small enough, and (2.7), (2.8), and (2.9) then give $f_1(z) = \prod_1(-1/a_j) z^{n(r, 1/F)}(1 + \rho(z))$, where $|\rho(z)| < \varepsilon$ for z satisfying (2.6). Estimating f_2 in the same way gives (2.2) and the proof of (2.3) is identical.

The following is Lemma III of [4].

Lemma B. *Suppose that* g *is meromorphic in* $|z| \le R$, $1 < r < R$, *and that* $I(r)$ *is a measurable subset of* $[0, 2\pi]$ *of measure* $\mu(r)$. *Then*

$$
\frac{1}{2\pi} \int_{I(r)} \log^+ |g(re^{i\theta})|\, d\theta \le 11R(R - r)^{-1}\mu(r) \left(1 + \log + \frac{1}{\mu(r)}\right) T(R, g) .
$$

3. Proof of Theorem 1

Suppose that f is transcendental and meromorphic in the plane such that $T(r, f) = o(\log r)^2$ and f has only finitely many critical values. By the remark in the introduction we can assume that f has no Picard value. Let R_ν, S_ν, C_ν, D_ν, m_ν, n_ν be as in Lemma 2. We can assume that, as $\nu \to \infty$,

$$
|C_\nu| R_\nu^{m_\nu} \to \alpha, \qquad 1 \le \alpha \le +\infty, \tag{3.1}
$$

by taking a subsequence and replacing f by $1/f$, if necessary. We consider a number of cases.

Case 1. Suppose that $\alpha = +\infty$ and $m_\nu = 0$ for infinitely many ν.

Taking a further subsequence if necessary we can assume that

$$
100|C_{\nu-1}| R_{\nu-1}^{m_{\nu-1}} < |C_\nu| < 100^{-1}|C_{\nu+1}| R_{\nu+1}^{m_{\nu+1}} . \tag{3.2}
$$

Take a small positive ε. Now (3.2) implies that if ν is large enough, the circle $|z| = R_\nu$ lies inside a bounded component of the set $\{z : |f(z) - C_\nu| < \varepsilon |C_\nu|\}$. This component contains no multiple point of f and is multiply connected, by (3.2), which contradicts the Riemann-Hurwitz formula.

Case 2. Suppose that $\alpha = +\infty$ and that $m_\nu \neq 0$ for all large ν.

Then (2.2) implies that the annulus $(1/4)R_\nu < |z| < 4R_\nu$ contains a closed level curve Γ_ν on which $|F(z)| = k_\nu = |C_\nu|R_\nu^{m_\nu}$, and this level curve Γ_ν must be a simple closed curve winding once around the origin. We take $\mu < \nu$ such that $100k_\mu < k_\nu$ and such that the region U lying between Γ_μ and Γ_ν contains at least one zero of f.

Let V_1 be a component of the set $\{z \in U : |f(z)| < k_\mu\}$. Since $|f(z)| \geq k_\mu$ on Γ_μ and Γ_ν, we have $|f(z)| = k_\mu$ on the boundary ∂V_1, which is contained in the closure $\overline{U}$ of U and consists of disjoint smooth simple closed curves. Let γ_1 be the unique component of ∂V_1 which forms the boundary of an unbounded component of $\mathbb{C}\backslash V_1$, and suppose first that the winding number $n(\gamma_1, 0) = 0$.

Now γ_1 cannot coincide with Γ_μ, since the interior of Γ_μ is bounded, and so γ_1 does not meet Γ_μ, using the fact that f has no critical values on $|w| = k_\mu$. Thus γ_1 forms part of the boundary of a component V^* of the set $\{z \in U : K_\mu < |f(z)| < k_\nu\}$. On ∂V^* we have $|f(z)| = k_\mu$ or $|f(z)| = k_\nu$, and V^* must be doubly-connected, by the Riemann-Hurwitz formula. The other component γ_2 of ∂V^* must close in $\overline{U}$ and cannot coincide with Γ_ν, since V^* is doubly-connected and since there exist points arbitrarily close to Γ_μ at which $|f(z)| < k_\mu$. Thus γ_2 cannot meet Γ_ν. Further, on γ_2 we have $|f(z)| = k_\nu$, and γ_2 forms part of the boundary of a component V^{**} of the set $\{z \in U : |f(z)| > k_\nu\}$, on the boundary of which $|f(z)| = k_\nu$. The Riemann-Hurwitz formula now implies that V^{**} is simply-connected, which is a contradiction, since V^* lies in a bounded component of $\mathbb{C}\backslash V^{**}$.

This contradiction proves that $n(\gamma_1, 0) \neq 0$. Thus Γ_μ lies in a bounded component of $\mathbb{C}\backslash V_1$, and ∂V_1 must have precisely two components ω_1 and ω_2 such that $n(\omega_j, 0) \neq 0$, and this is true for every component of the set $\{z \in U : |f(z)| < k_\mu\}$. If these components are $V_1, \dots, V_p$ and if $p > 1$, we can assume that, for each j, V_j lies in the same component of $\mathbb{C}\backslash V_{j+1}$ as Γ_μ. But then components of ∂V_1 and ∂V_2 together bound a doubly-connected region on which $|f(z)| \geq k_\mu$. This region must contain a pole of f by the maximum principle, and the fact that it is not simply-connected contradicts the Riemann-Hurwitz formula. Therefore $p = 1$, which is a contradiction since we can choose ν arbitrarily large.

Case 3. Suppose that α is finite in (3.1).

If $m_\nu \neq 0$ for infinitely many ν, then since $S_\nu \to \infty$ we can take a subsequence and obtain level curves Γ_ν on which $|f(z)| = k_\nu \to \infty$, by considering $f(z)$ on $|z| = R_\nu S_\nu^{\pm 1/4}$, and then argue as in Case 2. We assume henceforth that $m_\nu = 0$ for all large ν, so that without loss of generality

$$f(z) = 1 + o(1), \qquad R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu. \tag{3.3}$$

We also have (2.3), which we write in the form

$$f'(z) = D_\nu z^{n_\nu}(1 + \delta(z)), \qquad \delta(z) = o(1), \qquad R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu, \tag{3.4}$$

and we can assume that $\delta'(z) = o(1/|z|)$ for the same range of values of z, because otherwise we can replace S_ν by $S_\nu^{1/2}$ and apply Cauchy's estimate for derivatives.

Now if $n_\nu \le -2$ in (3.4), then integration by parts gives, with $q_\nu = n_\nu + 1$, $E_\nu = D_\nu q_\nu^{-1}$, $z_0 = R_\nu S_\nu$, and L_ν a constant, the estimates

$$(3.5)\quad f(z) = L_\nu + E_\nu z^{q_\nu}(1+\delta(z)) - \int_{z_0}^{z} E_\nu t^{q_\nu}\delta'(t)\,dt = L_\nu + E_\nu z^{q_\nu}(1+o(1)).$$

In obtaining the last estimate of (3.5) we have taken the path of integration to be the straight line segment from z_0 to $|z|$, followed by part of the circle $|t| = |z|$.

If $n_\nu = -1$ in (3.4), then the integral of $f'(z)$ around the circle $|z| = R_\nu$ will be $D_\nu(2\pi i + o(1))$, which is clearly impossible. Finally if $n_\nu \ge 0$ in (3.4), we take $z_0 = R_\nu S_\nu^{-1}$ and obtain (3.5) again.

Again we consider cases.

Case A. Suppose that $|1 - L_\nu| \ge (1/4)|E_\nu|R_\nu^{q_\nu}$ for infinitely many ν.

In this case, since $q_\nu \ne 0$ we find, using (3.3), that

$$f(z) - 1 = (L_\nu - 1)(1 + o(1)) = o(1)$$

either on $R_\nu S_\nu^{-1} \le |z| \le R_\nu S_\nu^{-1/2}$ or on $R_\nu S_\nu^{1/2} \le |z| \le R_\nu S_\nu$, and we can apply the reasoning of Case 1 to $g(z) = 1/(f(z) - 1)$.

Case B. Suppose that $|1 - L_\nu| < (1/4)|E_\nu|R_\nu^{q_\nu}$ for all large ν.

Again, since $q_\nu \ne 0$, we can obtain, on a smaller annulus formed as in Case A, the estimate $f(z) - 1 = E_\nu z^{q_\nu}(1 + o(1))$, and on these annuli $E_\nu z^{q_\nu} \to 0$ uniformly, by (3.3). Thus we may apply the reasoning of Case 2 to $g(z) = 1/(f(z) - 1)$.

4. Proof of Theorem 2

Let f_1 and f_2 be linearly independent rational solutions of (1.1), so that the Wronskian $W(f_1, f_2) = W$ is also rational. Now $(f_2/f_1)' = W f_1^{-2} = dz^{q-1}(1 + o(1))$ as $z \to \infty$, for some nonzero constant d and integer q, and q cannot be zero, since f_2/f_1 is by assumption rational. Therefore we may assume that $f_2(z)/f_1(z) = z^q(1 + o(1))$ as $z \to \infty$ and that q is positive.

Assuming that f and F are as in the statement of Theorem 2, and that $N(r, 1/fF) = o(T(r, f'/f))$ and f'/f is transcendental, we set

$$(4.1)\qquad H(z) = K_1(z)/K_2(z), \qquad K_j(z) = f_j'(z) - f_j(z)f'(z)/f(z),$$

so that H is transcendental of finite lower order.

Now all but finitely many poles of H are zeros of K_2 which are not zeros or poles of f. Further, $K_j'(z) = -f_j(z)F(z)/f(z) - K_j(z)(a_1(z) + f'(z)/f(z))$, so that at a zero z of K_2 with z large and with multiplicity $m \ge 2$, $F(z)$ must have a zero of multiplicity $m - 1$. Thus $N(r, H) - \overline{N}(r, H) \le N(r, 1/F) + O(\log r) = o(T(r, H))$, using (4.1). Moreover,

$$H'(z) = -W(z)F(z)/f(z)K_2(z)^2,$$

so that zeros z of H' with z large can only occur at zeros of F or at simple zeros of f, which implies that $N(r, 1/H') = o(T(r, H))$. We may therefore apply Theorem A to $g(z)$, where $g(z)$ is either $H(z)$ or $1/(b - H(z))$, for some constant b, g being normalized so that $\delta(\infty, g) = 0$.

We take a small positive constant ε and a sequence (r_k) such that r_0 is large and $2r_k \le r_{k+1} \le 4r_k$ for each $k \ge 0$ and such that the circles $|z| = r_k$ do not meet the exceptional set C_0 of Theorem A and further such that $T(r_k, f'/f) \le O(T(r_k, f))$ for each k. Now $L_2(r) = L_2(r_k) + o(1)$, uniformly for $r_k \le r \le r_{k+1}$. For each integer $k \ge 0$ we choose θ_k^* in $[L_2(r_k) - \pi/16\rho, L_2(r_k) + \pi/16\rho]$ such that the straight line segments $z = r\exp(i(\theta_k^* + j\pi/\rho))$, $r_k \le r \le r_{k+1}$, $0 \le j \le 2\rho - 1$, do not meet C_0. Obviously, $|\cos(\rho(\theta_k^* - L_2(r_k)))| \ge 3/4$. For each integer j with $0 \le j \le 2\rho - 1$ we then choose Γ_j to be the union of the straight line segments $z = r\exp(i(\theta_k^* + j\pi/\rho))$, $r_k \le r \le r_{k+1}$, $k \ge 0$, and the arcs $z = r_k\exp(i\theta)$, $|\theta - L_2(r_k) - j\pi/\rho| \le \pi/2\rho - \varepsilon$. On Γ_j, Theorem A gives $|g'(z)| \le |z|^{-3N}$, where N is a large positive integer, so $|g(z) - A_j| = O(|z|^{-2N})$, for some constant A_j. In fact a much stronger estimate is proved in [5], but this suffices for our purposes here and gives either $|H(z) - B_j| = O(|z|^{-2N})$, for some constant B_j, or $1/H(z) = O(|z|^{-2N})$. Thus either $G(z) = H(z)f_2(z)/f_1(z)$ or $G(z)^{-1}$ is $O(1/|z|)$ on Γ_j, and in either case we obtain there $f'(z)/f(z) = O(1/|z|)$, so $\log^+|1/f(z)| = O(\log|z|)$. Now Lemma B implies that for some small constant δ, which satisfies $\delta = O(\varepsilon\log(1/\varepsilon))$, we have, for each $k \ge 0$,

$$\begin{aligned} T(r_k, f) &\le (1 + o(1))m(r_k, 1/f) \\ &\le (\delta/2)T(2r_k, 1/f) + O(\log r_k) \le \delta T(r_{k+1}, f), \end{aligned}$$

so, for some positive constant c, independent of δ, and for $r_k \le r \le r_{k+1}$,

$$T(r, f) \ge T(r_k, f) \ge \delta^{-k}T(r_0, f) \ge \delta^{-2c\log r_k}T(r_0, f) \ge \delta^{-c\log r}T(r_0, f),$$

which contradicts the assumption that f has finite lower order and proves Theorem 2.

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