PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

ALMOST PERIODIC SOLUTIONS OF PERIODIC SYSTEMS GOVERNED BY SUBDIFFERENTIAL OPERATORS

MITSUHARU ÔTANI

(Communicated by Palle E. T. Jorgensen)

Dedicated to Professor Shige Toshi Kuroda on the occasion of his sixtieth birthday

ABSTRACT. We construct an example of the periodic evolution system governed by the time-dependent subdifferential operators admitting almost periodic orbits which are not quasiperiodic.

INTRODUCTION

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and $\Phi(H)$ be the set of all lower semicontinuous convex functions φ from H into $(-\infty, +\infty]$ with $D(\varphi) = \{u \in H; \varphi(u) < +\infty\} \neq \varnothing$. For a T-periodic mapping $t \longmapsto \varphi^t$ from $\mathbb{R}^1$ into $\Phi(H)$, we consider the following nonlinear evolution equation:

$$\text{(E)} \qquad \frac{d}{dt}u(t) + \partial\varphi^t(u(t)) \ni 0, \qquad t \in \mathbb{R}^1,$$

where $\partial\varphi^t$ is the subdifferential of φ^t (see Brézis [3]), i.e.,

$$\partial\varphi^t(u) = \{f \in H;\ \varphi^t(v) - \varphi^t(u) \geq (f, v-u)\ {}^\forall v \in D(\varphi^t)\}.$$

The main concern here is the structure of

$$\mathscr{R}\mathscr{C} = \{u : \mathbb{R}^1 \mapsto H;\, u \text{ satisfies (E) for a.e. } t \in \mathbb{R}^1 \text{ and } u(\mathbb{R}^1) \text{ is precompact in } H\},$$

the set of all precompact orbits of (E). For the case where $H = \mathbb{R}^d$, the properties of precompact (bounded) orbits has been one of the main objects of study in the theory of ordinary differential equations and dynamical systems. For $d = 2$ the classical Poincaré-Bendixson result asserts that the almost periodic solutions of $d\mathbf{x}(t)/dt = \mathbf{F}(\mathbf{x}(t))$ must be periodic, where $\mathbf{F}(\cdot)$ need not be monotone nor of subdifferential type. This kind of simple structure of the set of almost periodic orbits is peculiar to the case $d = 2$. In fact, Cartwright [4] showed that almost periodic orbits in $\mathbb{R}^d$ $(d > 2)$ are quasiperiodic with at most $d - 1$

Received by the editors July 19, 1993 and, in revised form, October 14, 1993.
1991 *Mathematics Subject Classification*. Primary 47H20.
Key words and phrases. Almost periodic orbit, periodic system, subdifferential operator.
The author was supported by a Waseda University Grant for Special Research Projects.

basic frequencies and moreover if $\mathbf{F}$ depends on t periodically, then the basic frequencies are at most d. This result was generalized by Cartwright [5] and O'Brien [12] to the case where $\mathbf{F}$ depends on t almost periodically, where they showed that the number of the basic frequencies in addition to those of $\mathbf{F}(t, \cdot)$ is still at most $d-1$. On the other hand if the system is governed by time-independent subdifferential operators, it was revealed by Baillon and Haraux [2] that $\mathscr{RC}$ has a very simple structure independent of the dimension of H. They studied the case where $\partial\varphi^t(u) = \partial\varphi(u) + f(t)$ and showed that every bounded solution of (E) on $\mathbb{R}^1$ must be T-periodic and the difference of any two T-periodic solutions is a constant (in time) vector. However for the general case where $\varphi^t(\cdot)$ depends on t, the structure of $\mathscr{RC}$ recovers its complexity. In fact, Kenmochi and Ôtani [10] constructed an example of a periodic system in $H = \mathbb{R}^3$ whose precompact orbits are not necessarily periodic (but quasiperiodic). Furthermore Haraux and Ôtani [7] showed that if $H = \mathbb{R}^d$, then any bounded orbit of (E) is quasiperiodic with at most $\left[\frac{d+1}{2}\right] = \max\{r \in \mathbb{N};\, r \leq \frac{d+1}{2}\}$ basic frequencies. This upper estimate for the number of basic frequencies is shown to be best possible by giving an example of a periodic system in $H = \mathbb{R}^d$ whose precompact orbits are quasiperiodic with $\left[\frac{d+1}{2}\right]$ basic frequencies. They also treated the case where $\partial\varphi^t$ is replaced by time-dependent general maximal monotone operators to assert that the same conclusion as above holds true with d replaced by $d+1$. Roughly speaking, almost periodic functions which are not quasiperiodic must have infinitely many basic frequencies. Thus under these observations it seems to be a natural question to ask whether there exists a periodic system in an infinite-dimensional Hilbert space which allows almost periodic but not quasiperiodic precompact orbits or not. As a matter of course it is well known that some nonparabolic P.D.E (such as wave equations) may have almost periodic solutions with infinitely many basic frequencies (see, e.g., Amerio and Prouse [1]). However these are described by the systems governed by maximal monotone operators which are not of subdifferential type. The main purpose of this paper is to give an affirmative answer to this question for the systems governed by subdifferential operators by constructing such an example in ℓ^2.

PRECOMPACT ORBITS IN ℓ^2

In this section, we are going to construct precompact orbits which are almost periodic but not quasiperiodic in ℓ^2. Denote the generic point x of ℓ^2 by $x = (x_0, x_1, \cdots, x_n, \cdots)$ with norm $|x| = \left(\sum_{j=0}^{\infty} x_j^2\right)^{1/2}$, and prepare two subspaces X_1 and X_0 defined by

$$X_1 = \left\{ x \in \ell^2;\ |x| \triangleq \left(x_0^2 + \sum_{k=1}^{\infty} 4^k (x_{2k-1}^2 + x_{2k}^2)\right)^{1/2} < +\infty \right\},$$

$$X_0 = \{x \in X_1;\ x_0 = 0\}.$$

For each $k \in \mathbb{N}$, $\theta \in [0, \pi)$, and $t \in [0, 1/2^k]$, we introduce an operator $R_k(\theta, t)$ which maps X_0 into X_1 by

$$x = (0, x_1, \cdots, x_n, \cdots) \in X_0,$$

where

$$(x_{2k-1}, x_{2k}) = r_k\ (\cos\alpha_k, \sin\alpha_k) \qquad (0 \le \alpha_k < 2\pi)$$

and $r_k = (x_{2k-1}^2 + x_{2k}^2)^{1/2}$:

$$R_k(\theta, t)\ x = (x_0(t), x_1(t), \cdots, x_n(t), \cdots)$$

with

$$\begin{cases} x_0(t) & = r_k\ \sin 2^k\pi t\ \sin(\theta - \alpha_k), \\ x_{2k-1}(t) & = r_k\ \left[\cos\theta\cos(\theta-\alpha_k) + \sin\theta\ \sin(\theta-\alpha_k)\cos 2^k\pi t\right], \\ x_{2k}(t) & = r_k\ \left[\sin\theta\cos(\theta-\alpha_k) - \cos\theta\ \sin(\theta-\alpha_k)\cos 2^k\pi t\right], \\ x_j(t) & = x_j, \quad j \ne 0, 2k-1, 2k. \end{cases}$$

The operation $(0, x_{2k-1}, x_{2k}) \mapsto (x_0(t), x_{2k-1}(t), x_{2k}(t))$ geometrically means the axial rotation of $(0, x_{2k-1}, x_{2k})$ in $\mathbb{R}^3$ with axis $\ell_\theta = \{(\xi_0, \xi_{2k-1}, \xi_{2k}) \in \mathbb{R}^3; \xi_0 = -\xi_{2k-1}\sin\theta + \xi_{2k}\cos\theta = 0\}$ and angle $2^k\pi t$. Hence the following properties are direct consequences of the definition of $R_k(\theta, t)$:

(1) $R_k(\theta, t)$ is a linear isometry from X_0 onto $R_k(\theta, t)\ X_0$, which forms a hyperplane in X_1, for each $k \in \mathbb{N}$, $\theta \in [0, \pi)$, and $t \in [0, 1/2^k]$.

(2) $R_k(\theta, 1/2^k)\ X_0 = X_0$.

(3) $R_k(\theta, t)$ is a C^∞-function of $t \in \mathbb{R}^1$ for any $k \in \mathbb{N}$ and $\theta \in [0, \pi)$.

For each $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_k, \cdots)$ with $0 < \theta_k < \pi$ $(k \in \mathbb{N})$, by composing $R_k(\cdot, t)$ we construct the operators $S_k(\theta_k, t)$ $(0 \le t \le 2^{-k+1})$ and $S(\mathbf{\Theta}, t)$ $(0 \le t < 2)$ as follows:

$$S_k(\theta_k, t) = \begin{cases} R_k(0, t), & 0 \le t \le 1/2^k, \\ R_k(\theta_k, t - 1/2^k)\ R_k(0, 1/2^k), & 1/2^k \le t \le 2/2^k, \end{cases} \tag{4}$$

$$S(\mathbf{\Theta}, t) = S_k(\theta_k, t - t_{k-1}) \prod_{j=1}^{k-1} S_j(\theta_j, 2^{1-j}) \quad {}^\forall t \in [t_{k-1}, t_k], k \in \mathbb{N}, \tag{5}$$

where $t_0 = 0$ and $t_k = \sum_{j=1}^k 2^{1-j} = 2(1 - 2^{-k})$. Then it is easy to see that

$$\begin{aligned} S_k(\theta_k, 2/2^k)x = (0, x_1, x_2, \cdots, x_{2k-1}, \\ r_k\cos(\alpha_k + 2\theta_k) r_k \sin(\alpha_k + 2\theta_k), x_{2k+1}, \cdots), \end{aligned}$$

whence follows

$$\begin{aligned} \lim_{t\to 2-0} S(\mathbf{\Theta}, t)\ x = (0, r_1\cos(\alpha_1 + 2\theta_1), r_1\sin(\alpha_1 + 2\theta_1), \cdots, \\ r_n\cos(\alpha_n + 2\theta_n), r_n\sin(\alpha_n + 2\theta_n), \cdots), \end{aligned} \tag{6}$$

where $x = (0, r_1\cos\alpha_1, r_1\sin\alpha_1, \cdots, r_n\cos\alpha_n, r_n\sin\alpha_n, \cdots)$.

Thus we can extend $S(\mathbf{\Theta}, t)$ up to $[0, 2]$ and further to the whole line $\mathbb{R}^1$ through the formula:

$$\begin{cases} S(\mathbf{\Theta}, t) = S(\mathbf{\Theta}, t - n)\ \left[S(\mathbf{\Theta}, 2)\right]^n, & 2n \le t \le 2(n+1),\ n \in \mathbb{N}, \\ \left[S(\mathbf{\Theta}, 2)\right]^0 = \mathrm{Id}, \quad \left[S(\mathbf{\Theta}, 2)\right]^n = \left\{\left[S(\mathbf{\Theta}, 2)\right]^{-1}\right\}^{-n}, & n \in \mathbb{Z},\ n < 0. \end{cases}$$

Let $X(t)$ be the image of X_0 under $S(\mathbf{\Theta}, t)$, i.e., $X(t) = S(\mathbf{\Theta}, t)\ X_0$. Then we have the following proposition.

Proposition 1. *The following properties hold.*

(i) $X(t)$ *has a period* 2, *i.e.*,

$$X(t+2n) = X(t), \quad {}^{\forall}n \in \mathbb{Z}, \quad {}^{\forall}t \in \mathbb{R}^1; \tag{7}$$

in particular $X(2n) = X_0$, ${}^{\forall}n \in \mathbb{Z}$.

(ii) $S(\Theta, t)$ *is a linear isometry from* X_0 *onto* $X(t)$, *and* $S(0)$ *is the identity on* X_0.

(iii) *For all* $x = (0, r_1 \cos\alpha_1, r_1 \sin\alpha_1, \cdots, r_n \cos\alpha_n, r_n \sin\alpha_n, \cdots)$,

$$\begin{aligned} &S(\Theta, 2m)x \\ &\quad = (0, r_1\cos(\alpha_1 + 2m\theta_1), r_1 \sin(\alpha_1 + 2m\theta_1), \cdots, \\ &\qquad\qquad r_n \cos(\alpha_n + 2m\theta_n), r_n \sin(\alpha_n + 2m\theta_n), \cdots). \end{aligned} \tag{8}$$

(iv) *The right and left derivatives* $(d^+/dt)S(\Theta, t)x$, $(d^-/dt)S(\Theta, t)\,x$ *exist for every* $t \in \mathbb{R}^1$ *and* $x \in X_0$. *Moreover*

$$(d^{\pm}/dt)S(\Theta, t)x \in X(t)^{\perp} = \text{the orthogonal complement of } X(t) \text{ in } \ell^2, \tag{9}$$

$$\sup_t \left|\frac{d}{dt}S(\Theta, t)x\right|_{\ell^2} \le \left(\sum_{k=1}^{\infty} 4^k\, r_k^2\right)^{1/2} \pi = |x|_{X_1}\, \pi < +\infty \quad {}^{\forall}x \in X_0. \tag{10}$$

Proof. It is clear that properties (i)-(iii) follow from the facts (1)-(3) and (6). Recalling the geometrical meaning of $R_k(\theta, t)$ (or by the direct calculation), we can easily deduce (9). Moreover, since $|\frac{d}{dt}(R_k(\theta, t)x)|_{\ell^2} = r_k 2^k \pi |\sin(\theta - \alpha_k)|$, we obtain

$$\sup_t \left|\frac{d}{dt}S(\Theta, t)x\right|_{\ell^2} \le \sup_k r_k\; 2^k\; \pi \le \left(\sum_{k=1}^{\infty} r_k^2\; 4^k\right)^{1/2} \pi = |x|_{X_1}\pi. \quad \square$$

The compactness of orbits $S(\Theta, t)x$ is assured by the following result.

Proposition 2. *Let* $X_1^R = \{x \in \ell^2; |x|_{X_1}^2 = x_0^2 + \sum_{k=1}^{\infty} 4^k(x_{2k-1}^2 + x_{2k}^2) \le R^2\}$, $R > 0$; *then* X_1^R *is a compact set in* ℓ^2.

In particular, $X_0^R = \{x \in X_0\ ;\ |x|_{X_1}^2 \le R^2\}$ *and* $X^R(t) \triangleq S(\Theta, t)\; X_0^R$ *are compact sets in* ℓ^2.

Proof. Let $x^n = (\xi_0^n, \xi_2^n, \cdots, \xi_k^n, \cdots) \in X_1^R$; then by the usual diagonal argument we can extract a subsequence of x^n denoted again by x^n such that $\xi_k^n \to \xi_k$ as $n \to \infty$ for all $k \in \mathbb{N} \cup \{0\}$. Since

$$\left\{(\xi_0^n)^2 + \sum_{k=1}^{m} 4^k \left((\xi_{2k-1}^n)^2 + (\xi_{2k}^n)^2\right)\right\}^{1/2} \le R \quad \text{for all } m \in \mathbb{N}$$

and $(\xi_0^n, \xi_1^n, \cdots, \xi_m^n)$ converges to $(\xi_0, \xi_1, \cdots, \xi_m)$ in $\mathbb{R}^m$, we easily see

$$\left\{(\xi_0)^2 + \sum_{k=1}^{m} 4^k(\xi_{2k-1}^2 + \xi_{2k}^2)\right\}^{1/2} \le R \quad \text{for all } m \in \mathbb{N},$$

whence follows $x = (\xi_0, \xi_1, \cdots, \xi_m, \cdots) \in X_1^R$. Moreover $x \in X_1^R$ implies that $(\xi_{2k-1}^2 + \xi_{2k}^2)^{1/2} \le 2^{-k} R$ for all $k \in \mathbb{N}$ and that for an arbitrary $\varepsilon > 0$

there exists a number N such that $2\sum_{k=N+1}^{\infty}(2^{-k}R)^2 < \varepsilon^2$. Therefore

$$|x^n - x|_{\ell^2}^2 \le \sum_{j=0}^{2N} |\xi_j^n - \xi_j|^2 + \sum_{j=2N+1}^{\infty} |\xi_j^n|^2 + \sum_{j=2N+1}^{\infty} |\xi_j|^2$$
$$< \sum_{j=0}^{2N} |\xi_j^n - \xi_j|^2 + \varepsilon^2.$$

Hence $\limsup_{n\to\infty} |x^n - x|_{\ell^2} \le \varepsilon$ for all $\varepsilon > 0$, i.e., $x^n \to x$ in ℓ^2. Thus X_1^R is proved to be compact in ℓ^2. The same argument as above assures that X_0^R is also compact in ℓ^2. Since $X^R(t)$ is the image of X_0^R by $S(\Theta, t)$ which is continuous from ℓ^2 into itself, $X^R(t)$ turns out to be compact in ℓ^2 for all $t \in \mathbb{R}^1$. $\square$

Periodic systems

Let R be a positive number and for each t we put

$$\varphi^t(x) = \begin{cases} 0 & \text{if } x \in X^R(t), \\ +\infty & \text{if } x \in \ell^2 \setminus X^R(t). \end{cases}$$

Since $X^R(t)$ is closed convex, it is clear that $\varphi^t \in \Phi(\ell^2)$, $D(\varphi^t) = X^R(t)$ and $X(t)^\perp \subset \partial\varphi^t(x)$ for any $x \in X^R(t)$. Furthermore, in view of (7), (9) and (10), we find that φ^t is periodic with period 2 and for every $x \in X_0^R$, $u(t) = S(\Theta, t)\, x$ gives a solution of the equation (E) with $u(0) = x$. At the same time, this fact assures that $\{\varphi^t\}_{t\in\mathbb{R}^1}$ satisfies the *t-dependence* conditions introduced in Yamada [15], Kenmochi [8,9], Kenmochi and Ôtani [11], Haraux and Ôtani [7] and Ôtani [13,14].

Now the existence of an almost periodic solution of (E) which is not quasiperiodic is assured by the following result. For the notion of almost periodicity and quasiperiodicity, we refer to Amerio and Prouse [1] and Fink [6].

Theorem. *Let* $\Theta = (\theta_1, \theta_2, \cdots, \theta_n, \cdots)$ *be linearly independent over* $\mathbb{Q}$, *i.e., for every* $m \in \mathbb{N}$, $\{\theta_1, \cdots, \theta_m\}$ *are linearly independent over* $\mathbb{Q}$. *Then for every* $x \in X_0^R$, $u(t, x) = S(\Theta, t)\, x$ *gives a solution of* (E), *which is almost periodic but not quasiperiodic.*

Proof. The precompactness of the orbit $\{S(\Theta, t)\, x\}_{t\in\mathbb{R}^1}$ is insured by Proposition 2. Therefore it follows from Theorem I of [7] that $u(t, x)$ is almost periodic. Let $x = (0, r_1\cos\alpha_1, r_1\sin\alpha_1, \cdots, r_n\cos\alpha_n, r_n\sin\alpha_n, \cdots) \in X_0^R$ and e_j be the unit vector whose j-th component is 1.

Since $x = \sum_{k=1}^{\infty}(r_k\cos\alpha_k\, e_{2k-1} + r_k\sin\alpha_k\, e_{2k})$, we get

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t),$$

where $u_k(t, x) = S(\Theta, t)\, (r_k\cos\alpha_k\, e_{2k-1} + r_k\sin\alpha_k\, e_{2k})$.

For any $m \in \mathbb{N}$, define the restriction operator P_m from ℓ^2 onto $\mathbb{R}^{2m+1}$ by $x = (x_0, x_1, \cdots, x_k, \cdots) \mapsto P_m x = (x_0, x_1, \cdots, x_{2m+1})$. Put

$$\varphi_m^t(x) = \begin{cases} 0 & \text{if } x \in X_m^R(t) \triangleq P_m X^R(t), \\ +\infty & \text{if } x \in \mathbb{R}^{2m+1} \setminus X_m^R(t). \end{cases}$$

Then $\varphi_m^t \in \Phi(\mathbb{R}^{2m+1})$ and φ_m^t has a period 2. Furthermore $v^m(t) \triangleq P_m u(t, x) = \sum_{k=1}^m P_m u_k(t)$ satisfies: $dv^m(t)/dt + \partial\varphi_m^t(v^m(t)) \ni 0$. Therefore as is shown in Remark 4.3 of [7], $v^m(t)$ is quasiperiodic with $(m+1)$ basic frequencies $\{2\pi, \theta_1, \cdots, \theta_m\}$. Hence $\sum_{k=1}^m u_k(t)$ (the zero extension of $v^m(t)$ to ℓ^2) is also quasiperiodic with $(m+1)$ basic frequencies. Thus $u(t, x) = \sum_{k=1}^\infty u_k(t)$ has a countable number of basic frequencies $\{2\pi, \theta_1, \cdots, \theta_m, \cdots\}$, so can not be quasiperiodic. □

Acknowledgment

The author would like to thank the referee for informing him of references [4, 5, 12].

References

1. L. Amerio and G. Prouse, *Almost periodic functions and functional equations*, Van Nostrand, New York, Cincinnati, Toronto, London, and Melbourne, 1971.
2. J. B. Baillon and A. Haraux, *Comportement à l'infini pour les équations d'évolution avec forcing périodique*, Arch. Rational Mech. Anal. **67** (1977), 101–109.
3. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.
4. M. L. Cartwright, *Almost periodic flows and solutions of differential equations*, Proc. London Math. Soc. (3) **17** (1967), 355–380; corrigenda **17** (1967), 768.
5. ——, *Almost periodic differential equations and almost periodic flows*, J. Differential Equations **5** (1969), 167–181.
6. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Math., vol. 377, Springer-Verlag, Berlin, Heidelberg, and New York, 1974.
7. A. Haraux and M. Ôtani, *Quasi-periodicity of bounded solutions to some periodic evolution equations*, J. Math. Soc. Japan **42** (1990), 277–294.
8. N. Kenmochi, *Some nonlinear parabolic variational inequalities*, Israel J. Math. **22** (1975), 304–331.
9. ——, *Solvability of nonlinear evolution equations with time-dependent constrains and applications*, Bull. Fac. Education Chiba Univ. **30** (1981), 1–87.
10. N. Kenmochi and M. Ôtani, *Instability of periodic solutions of some evolution equations governed by time-dependent subdifferential operators*, Proc. Japan Acad. **61** (1985), 4–7.
11. ——, *Asymptotic behavior of periodic systems generated by time-dependent subdifferential operators*, Funkcial. Ekvac. **29** (1986), 219–236.
12. G. C. O'Brien, *The frequencies of almost periodic solutions of almost periodic differential equations*, J. Austral. Math. Soc. **17** (1974), 332–344.
13. M. Ôtani, *Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Periodic problems*, J. Differential Equations **54** (1984), 248–273.
14. ——, *Nonlinear evolution equations with time-dependent constraints*, Adv. Math. Sci. Appl. **3** (1993/94), 383–399.
15. Y. Yamada, *Periodic solutions of certain nonlinear parabolic differential equations in domains with periodically moving boundaries*, Nagoya Math. J. **70** (1978), 111–123.

Department of Applied Physics, School of Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo, Japan

E-mail address: otani@cfi.waseda.ac.jp