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BOUNDED POINT EVALUATIONS AND POLYNOMIAL APPROXIMATION

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ABSTRACT. We consider the set of bounded point evaluations for polynomials with respect to the L^p-norm for a measure. We give an example of a measure where the corresponding sets of bounded point evaluations vary with the exponent p. The main ingredient is the remarkable work of K. Seip on interpolating and sampling sequences for weighted Bergman spaces.

1. INTRODUCTION

For a positive measure μ with compact support in the complex plane and for $1 \leq t < \infty$ let $P^t(\mu)$ denote the closure in $L^t(\mu)$ of the analytic polynomials. A point w is a *bounded point evaluation* (bpe) for $P^t(\mu)$ if there exists a constant M such that $|p(w)| \leq M\|p\|$ for every polynomial p. In [8] we describe $P^t(\mu)$ and establish the existence of a large open set of bpes if $P^t(\mu) \neq L^t(\mu)$. The purpose of this paper is to give examples of measure where the corresponding sets of bpes vary with the exponent t.

In all previously known examples the set of bpes is independent of the exponent t. For example, if μ is supported on the unit circle, then Szego's theorem implies that the set of bpes is determined by point masses and the Radon-Nikodym derivative of μ with respect to Lebesgue measure. If the derivative is log integrable, then the set of bpes includes the open unit disk . If not, then $P^t(\mu) = L^t(\mu)$ for all t and all bpes arise from point masses. However, if Lebesgue measure is absolutely continuous with respect to μ, then the set

$$\{f\,d\mu\colon f \in L^1(\mu)\}$$

includes all the Poisson kernels as measures; and hence point evaluations at points in the open unit disk are weak-star continuous. (Here we are considering the polynomials as a subset of $L^\infty(\mu)$, which has a weak-star topology.) It follows that there exists a measure μ on the circle with no bounded point evaluations for $t < \infty$ but with weak-star continuous evaluations.

Historically, polynomial approximation with respect to area measure on simply connected regions has been studied extensively. References to major results and examples can be found in [2] and [5]. More recently, John Akeroyd [1] has

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determined the set of bpes for $P^t(\mu)$ for a large class of crescents, where μ is harmonic measure on the boundary of the crescent.

Our examples are based on the remarkable work of Kristian Seip [6, 7]. Seip completely describes the interpolating and sampling sequences for weighted Bergman spaces. He also gives examples of sequences that lie on the edge between the two concepts. Let A denote normalized Lebesgue measure on the open unit disk U. Let a and b be positive numbers, and let μ be the measure with $d\mu = (1-|z|^2)^{2a-1}dA$. Seip constructs a sequence Γ in U (depending only on b) such that Γ is interpolating for $P^2(\mu)$ if $a > b$ and Γ is a set of sampling for $P^2(\mu)$ if $a < b$. This example leads to our examples.

Our first example is an atomic measure μ with the property that $P^t(\mu) = L^t(\mu)$ if $1 \le t < 2$ while $P^t(\mu)$ is a space of analytic functions if $t > 2$. For our second example σ we add a weighted area measure to μ. The set of bpes for $P^3(\sigma)$ is U, but the set of bpes for $P^1(\sigma)$ is $U\backslash[0, 1)$.

2. Background and Seip's theorems

Define for each $n > 0$ the weighted Bergman space $A^{-n,2}$ to be the Banach space of functions in $L^2((1-|z|^2)^{2n-1}dA)$ that are analytic in U. Observe that there exist positive constants c_j (depending on n) such that for $f(z) = \sum_j a_j z^j$ in $A^{-n,2}$

$$\|f\|^2 = (1/\pi)\iint |f|^2(1-r^2)^{2n-1} r\,dr\,d\theta = \sum_j c_j|a_j|^2 .$$

Thus, the partial sums of the Taylor series for f converge to f in norm. Consequently, $A^{-n,2} = P^2((1-|z|^2)^{2n-1}dA)$.

Now we follow Seip [7]. Let

$$\rho(z, w) = \left|\frac{z-w}{1-\overline{z}w}\right|$$

which is the pseudohyperbolic distance function on U. We say that a sequence $\Gamma = \{z_j\}$ is *uniformly discrete* (or separated) if

$$\inf_{j\neq k} \rho(z_j, z_k) > 0 .$$

For a uniformly discrete set $\{z_j\}$ and $\frac{1}{2} < r < 1$ let

$$D(\Gamma, r) = \frac{\sum \log \frac{1}{|z_j|}}{\log \frac{1}{1-r}}$$

where the sum is taken over all j with $\frac{1}{2} < |z_j| < r$. For each z in U we form a new sequence

$$\Gamma_z = \left\{\frac{z_j - z}{1-\overline{z}z_j}\right\} .$$

The *lower* and *upper uniform densities* of Γ are defined, respectively, as

$$D^-(\Gamma) = \liminf_{r\to 1} \inf_{z\in U} D(\Gamma_z, r)$$

and

$$D^+(\Gamma) = \limsup_{r\to 1} \sup_{z\in U} D(\Gamma_z, r) .$$

The key example of a sequence is the following sequence from Seip [7]. For $a > 1$, $b > 0$, let Γ denote the image of $\{a^j(bk+i)\}_{j,k\in Z}$ under the Cayley transform of the upper half-plane to U. Then

$$D^-(\Gamma) = D^+(\Gamma) = \frac{2\pi}{b\log a}.$$

The following relationship between atomic measures and area measure is an immediate consequence of [7, Equation (2)]. Let $\{z_j\}$ be a uniformly discrete sequence in U, and let $\delta = \inf_{j\neq k}\rho(z_j, z_k)$. Then for f analytic in U

$$\sum(1-|z_j|^2)^s|f(z_j)|^2 \le C(\delta)\int(1-|z|^2)^{s-2}|f(z)|^2\,dA(z) \tag{1}$$

whenever $s > 0$ (both sides may be infinite).

A sequence $\{z_j\}$ of distinct points in U is a *set of sampling* for $A^{-n,2}$ if there exist positive constants K_1 and K_2 such that

$$\begin{aligned}K_1\int|f|^2(1-|z|^2)^{2n-1}\,dA(z) &\le \sum|f(z_j)|^2(1-|z_j|^2)^{2n+1}\\ &\le K_2\int|f|^2(1-|z|^2)^{2n-1}\,dA(z)\end{aligned}$$

for every f in $A^{-n,2}$. The sequence $\{z_j\}$ is a *set of interpolation* for $A^{-n,2}$ if for every sequence $\{a_j\}$ for which $\sum(1-|z_j|^2)^{2n+1}|a_j|^2 < \infty$ there exists a function f in $A^{-n,2}$ such that $f(z_j) = a_j$ for all j.

We now state Seip's theorems for weighted Bergman spaces [7].

Theorem 2.1. *A sequence Γ of distinct points in U is a set of sampling for $A^{-n,2}$ if and only if it can be expressed as a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > n$.*

Theorem 2.2. *A sequence Γ of distinct points in U is a set of interpolation for $A^{-n,2}$ if and only if Γ is uniformly discrete and $D^+(\Gamma) < n$.*

3. Examples

Let $n > 0$, and let $\Gamma = \{z_j\}$ be a uniformly discrete sequence with $D^+(\Gamma) = D^-(\Gamma) = n$. Let $d = \inf_{j\neq k}\rho(z_j, z_k)$. For z in U let δ_z denote the measure of point mass at z. Let α be the sigma-finite measure $\sum\delta_{z_j}$.

Let μ be the measure with $d\mu = (1-|z|^2)^{2n+1}d\alpha$. Using (1) with $f\equiv 1$, we see that

$$\int(1-|z|^2)^{2n+1}\,d\alpha \le C(d)\int(1-|z|^2)^{2n-1}\,dA(z) < \infty.$$

Thus, the measure μ is finite. This argument also applies to each atomic measure introduced in the proof of the following theorem.

Theorem 3.1. *If $1\le t<2$, then $P^t(\mu) = L^t(\mu)$. If $t > 2$, then $P^t(\mu) \neq L^t(\mu)$.*

Proof. First consider the case where $1 \le t < 2$. Let $s = t/2$. Choose $m > n$ such that $sm < n$, and let $\varepsilon = (m-n)/(1-s)$. Let ν be the measure with $d\nu = (1-|z|^2)^{2m+1-2\varepsilon}d\alpha$.

Because $\varepsilon < m$, the measure ν is finite. Also the equality

$$2n+1 = 2m+1-2\varepsilon+2\varepsilon s$$

implies that

$$d\mu = (1 - |z|^2)^{\varepsilon t} d\nu .$$

Let τ be the measure with $d\tau = (1 - |z|^2)^{2\varepsilon} d\nu$.

Now let q be the positive number with $(1/q)+s = 1$. Let p be a polynomial. By Holder's inequality

$$\begin{aligned}\int |p|^t \, d\mu &= \int |p|^t (1 - |z|^2)^{\varepsilon t} \, d\nu \\ &\leq \left(\int |p|^2 (1 - |z|^2)^{2\varepsilon} \, d\nu \right)^s \|\nu\|^{1/q} .\end{aligned}$$

Hence there is a constant C, independent of the polynomial p, such that

$$\|p\|_{L^t(\mu)} \leq C \|p\|_{L^2(\tau)} . \tag{2}$$

Because $m > D^+(\Gamma)$, it follows from Theorem 2.2 that Γ is interpolating for $A^{-m,2}$. Using (1) and noting that $d\tau = (1 - |z|^2)^{2m+1} d\alpha$, we see that the characteristic function of each singleton is in $P^2(\tau)$. It now follows from (2) that each such characteristic function is in $P^t(\mu)$. Thus $P^t(\mu) = L^t(\mu)$.

Next consider the case where $t > 2$. Again let $s = t/2$. Choose m such that $0 < m < n$ and $n < ms$. Define ε and ν symbolically the same as in the previous case. Note that we again have $0 < \varepsilon < m$ and $d\mu = (1 - |z|^2)^{\varepsilon t} d\nu$. Let τ be the measure with $d\tau = (1 - |z|^2)^{2m-1} dA$.

Now let q be the positive number with $(1/q) + (2/t) = 1$. Let p be a polynomial. Because $m < D^-(\Gamma)$, it follows from Theorem 2.1 that Γ is a set of sampling for $A^{-m,2}$. Thus there exists a constant C such that

$$\begin{aligned}\int |p|^2 (1 - |z|^2)^{2m-1} \, dA &\leq C \int |p|^2 (1 - |z|^2)^{2m+1} \, d\alpha \\ &= C \int |p|^2 (1 - |z|^2)^{2\varepsilon} \, d\nu \\ &\leq C \int (|p|^t (1 - |z|^2)^{\varepsilon} \, d\nu)^{2/t} \|\nu\|^{1/q} .\end{aligned}$$

It now follows that there exists a positive constant K such that

$$\|p\|_{L^2(\tau)} \leq K \|p\|_{L^t(\mu)} .$$

Observing that each point in U is a bpe for $P^2(\tau)$, we see that U equals the set of bpes for $P^t(\mu)$. □

Remark. In the case where $P^t(\mu) = L^t(\mu)$ the set of bpes for $P^t(\mu)$ equals the set of atoms of μ. It is obvious that each atom gives rise to a bpe, so it suffices to consider a point λ that is not an atom. Since $(z - \lambda) L^t(\mu)$ is dense in $L^t(\mu)$, it follows that the polynomials that vanish at λ are dense in $L^t(\mu)$ also. In particular, the constant function one is in the closure of the set of polynomials that vanish at λ. But the constant function one takes on the value one at each bpe, so λ cannot be a bpe.

Theorem 3.2. *There exists a measure σ such that the set of bpes for $P^3(\sigma)$ equals U and the set of bpes for $P^1(\sigma)$ equals $U \setminus [0, 1)$.*

Proof. Let n, Γ, and μ be as indicated at the start of this section. Applying a Mobius transformation to Γ, if necessary, we may assume that Γ does not meet the interval $[0, 1)$.

Now let $t = 1$, and choose m and τ as in the first part of the proof of Theorem 3.1. Recall that Γ is an interpolating sequence for $A^{-m,2}$. Thus, there exists a nonconstant function f in $A^{-m,2}$ that vanishes on Γ. Let $u(z) = (1 - |z|^2)^{2m-1}$, so $P^2(udA) = A^{-m,2}$. Using (1), we see that each sequence of polynomials converging to f in $P^2(udA)$ also converges to f in $P^2(\tau)$. Now using (2), we see that each such sequence also converges to f in $P^1(\mu)$. Thus, f belongs to $P^1(d\mu + udA)$.

By a method of W. Hastings [4; 3, p. 83], there is a weight function w defined on $U\backslash[0, 1)$ such that (a branch of) $z^{1/2}$ belongs to $P^1(w|f|udA)$. Furthermore, we may assume that $0 < w \leq 1$ and w is bounded below on each compact subset of $U\backslash[0, 1)$.

Let σ be the measure with $d\sigma = d\mu + wudA$. Since u and w are bounded below on each compact subset of $U\backslash[0, 1)$, it follows that each point in $U\backslash[0, 1)$ is a bpe for $P^1(\sigma)$. Recalling that Γ does not meet $[0, 1)$, we may conclude that $P^1(\sigma)$ contains no nontrivial L^1-summand. By [8] it follows that the set of bpes is open and that each function in $P^1(\sigma)$ extends to be analytic on the set of bpes.

The function f above is in $P^1(\sigma)$ because $w \leq 1$. It follows from the defining property of w that $z^{1/2}f$ belongs to $P^1(\sigma)$. But $z^{1/2}f$ cannot be extended to be analytic in any region containing a point on $[0, 1)$. Thus, the set of bpes for $P^1(\sigma)$ equals $U\backslash[0, 1)$.

Since the set of bpes for $P^3(\mu)$ equals U, the same conclusion holds for $P^3(\sigma)$. $\square$

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