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REGULAR OPERATOR CONVERGENCE AND NONLINEAR EQUATIONS INVOLVING NUMERICAL RANGES

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ABSTRACT. Regular operator approximation theory, based on the work of Anselone and Lei (1986), is generalized to the case of strongly accretive operators and applied to nonlinear equations involving the generalized Zarantonello numerical ranges

1. INTRODUCTION

Recently Anselone and Lei [4] studied the regular operator approximation theory, which heavily relies on inverse compactness principles and provides a convenient general framework for the convergence of approximation solutions; then the existence of solutions follows in a natural manner. They applied this theory to the solvability of equations involving strongly monotone operators, and, as a result, they illustrated the theory to nonlinear integral equations of Urysohn and Hammerstein types. For a detailed account on regular approximation theory, we refer to [1–6, 15].

In this paper our aim is to generalize the results of Anselone and Lei [4] to the case of nonlinear equations involving strongly accretive operators in a slightly different setting. The obtained results are applied to the equations involving the theory of the generalized Zarantonello numerical ranges.

Now we need to recall some definitions. Let X and Y be two Banach spaces. Let $N = \{1, 2, \dots\}$ with infinite subsets $N', N'', \dots$. Let $\{x_n\} = \{x_n : n \in N\} \subset X$, and let $S_n \subset X$ for $n \in N$. Then $\{x_n\}^* = \{x \in X : x_n \to x, n \in N'\}$ and $\{S_n\}^* = \{x \in X : x_n \to x,\ x_n \in S_n,\ n \in N'\}$ are called the sets of all cluster points (limits of sequences).

A sequence $\{x_n\}$ is said to be *d-compact* (*discretely compact*) if every subsequence has a convergent subsequence. Alternatively, $\{x_n\}$ is d-compact if $\{x_n : n \in N'\}^* \neq \varnothing$ for $n \in N'$. Similarly, $\{S_n\}$ is called *d-compact* if every subsequence $\{x_n \in S_n : n \in N'\}$ has a convergent subsequence. Alternatively, $\{S_n\}$ is d-compact if $\{S_n : n \in N'\}^* \neq \varnothing$ for all N' such that $S_n \neq \varnothing$ for $n \in N'$. If $S_n = \varnothing$ for $n \in N$, then $\{S_n\}$ is trivially d-compact.

A set sequence $\{S_n\}$ is said to converge to a set S ($S_n \to S$ as $n \to \infty$) if any ε-neighboorhood of S contains S_n for all n sufficiently large. Such a

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limit is not unique, that is, $S_n \to S \subset S' \Rightarrow S_n \to S'$. Similarly, $S_n = \varnothing$ for $n \in N \Rightarrow S_n \to S$ for all $S \subset X$, $S_n \to \varnothing \Rightarrow S_n = \varnothing$ for n sufficiently large, and $S_n \neq \varnothing$ for $n \in N'$, $S_n \to S \Rightarrow S \neq \varnothing$. The connection between the set convergence and d-compactness is given by

$$\{S_n\} \ d\text{-compact and } \{S_n\}^* \subset S \Rightarrow S_n \to S.$$

For K, $K_n\colon X \to Y$, $K_n \to K$ represents *pointwise convergence*, that is, $K_n x \to Kx$ for $x \in X$ as $n \to \infty$, and $K_n \xrightarrow{c} K$ represents *continuous convergence*, that is, $x_n \to x$ implies $K_n x_n \to Kx$ for all $x \in X$.

A word of caution: here and in what follows, the symbols "$\to$" and "$\xrightarrow{w}$" shall denote, respectively, the strong convergence and the weak convergence.

2. Regular operator approximation theory

Let us consider operators A, $A_n\colon X \to Y$. An operator A is called *regular* if $\{x_n\}$ bounded and $\{Ax_n\}$ d-compact implies $\{x_n\}$ d-compact. An operator sequence $\{A_n\}$ is *asymptotically regular* if $\{x_n\colon n \in N'\}$ bounded and $\{A_n x_n\colon n \in N'\}$ d-compact implies $\{x_n\colon n \in N'\}$ d-compact. The *regular convergence* ($A_n \xrightarrow{r} A$) is defined as follows: $A_n \xrightarrow{r} A$ if $A_n \xrightarrow{c} A$ and $\{A_n\}$ is asymptotically regular. We note that $A_n \xrightarrow{c} A$ implies that A is continuous, and $A_n \xrightarrow{r} A$ implies that A is regular and continuous. By simple arguments we find that $\{A_n\}$ asymptotically regular and $|\lambda_n| \geq \delta > 0$ for all $n \geq n_0$ implies that $\{A_n \lambda_n\}$ is asymptotically regular, and $A_n \xrightarrow{r} A$ and $\lambda_n \to \lambda \neq 0$ implies that $\lambda_n A_n \xrightarrow{r} \lambda A$.

To include some more definitions, let X be a real Banach space with its dual X^* uniformly convex. For $n \in N$, let P_n be the projection on X with $P_n X = X_n$, $\dim X_n < \infty$, and $P_n \to I$ as $n \to \infty$. Then $\{P_n\}$ is uniformly bounded, $P_n \xrightarrow{c} I$, that is, $P_n x_n \to x$ as $x_n \to x$, and the convergence in $P_n \to I$ is uniform on a compact set. Let $J\colon X \to X^*$ be a normalized duality mapping, that is, $[x, Jx] = \|x\|^2$ and $\|Jx\| = \|x\|$, where $[\cdot, , \cdot]$ is the duality pairing between the elements of X and X^*. Here P_n^* is a projection on X^* with $P_n^* \to I^*$ as $n \to \infty$.

An operator $A\colon X \to X$ is said to be *strongly accretive* if, for a constant $\alpha > 0$,

$$[Ax - Ay, J(x-y)] \geq \alpha \|x-y\|^2 \quad \text{for all } x, y \in X. \tag{2.1}$$

We note that (2.1) also implies that

$$|[Ax - Ay, J(x-y)]| \geq \alpha \|x-y\|^2 \quad \text{for all } x, y \in X. \tag{2.2}$$

Next, we shall need to recall a theorem of Anselone and Ansorge [2] crucial to the work at hand. The scope of the theorem extends to the case of compact operators as well.

Lemma 2.1 [2, Theorem 4.10]. *Let X and Y be two Banach spaces and A, $A_n\colon X \to Y$ be operators from X to Y. If $A_n \xrightarrow{r} A$, $y_n \to y$, $\gamma > 0$, and*

$$S = \{x \in X\colon Ax = y,\ \|x\| \leq \gamma\}, \text{ and}$$
$$S_n = \{x_n \in X\colon A_n x_n = y_n,\ \|x_n\| \leq \gamma\},$$

then $\{S_n\}$ is d-compact, $\{S_n\}^* \subset S$, and $S_n \to S$.

Lemma 2.2 [11, Lemma 18.2]. *Let R^n be a real Euclidean space with the inner product $\langle \cdot , \cdot \rangle$. Assume $D \subset R^n$ is bounded, open, and convex, and $0 \in D$. Assume $F : \overline{D} \to R^n$ is continuous and $\langle Fx , x \rangle > 0$ for $x \in \delta D$. Then $Fx = 0$ for some $x \in D$.*

Lemma 2.3 [16, Proposition]. *Let X be a separable Banach space with X^* uniformly convex, and let $\{x_n\}$ be a bounded sequence in X. Then there exist a subsequence $\{x_k\}$ (say) and a point $v \in X$ such that $\{J(x_n - v)\} \xrightarrow{w} 0$ in X^*.*

Now we compare the equations

$$\begin{cases} Ax = b , & x , b \in X , \\ P_n A x_n = P_n b , & x_n \in X_n , \ b \in X . \end{cases}$$

Here

$$\begin{aligned} P_n A x_n = P_n b \Leftrightarrow & [P_n(Ax_n - b) , P_n^* J x] \\ & = [P_n(Ax_n - b) , Jx] \\ & = [Ax_n - b , P_n^* J x] \\ & = [Ax_n - b , Jx] = 0 \quad \text{for } x \in X_n . \end{aligned} \tag{2.3}$$

We are about to consider the main results.

Theorem 2.4. *Let $A : X \to X$ be bounded, continuous, and strongly accretive with a constant $\alpha > 0$. If P_n is projection on X with $X_n = P_n X$ and $J : X \to X^*$ is a normalized duality mapping, where X^* is uniformly convex, then the following implications hold:*

(i) *For $b \in X$, $\|Jx\| = \|x\| \geq \gamma > \|A(0) - b\|/\alpha$, $x \in X$, $\gamma > 0$,*

$$[Ax - b , Jx] > 0 .$$

(ii) *For $b \in X$, $\gamma > 0$, and $[Ax - b , Jx] > 0$ with $\|x\| = \gamma$, we have*

$$P_n A x_n = P_n b \quad \text{for some } x_n \in X_n \text{ with } \|x_n\| < \gamma , \ n \in N .$$

(iii) $P_n A \xrightarrow{r} A$.

(iv) *For $b \in X$ and $\gamma > \|A(0) - b\|/\alpha$,*

$$S_n = \{x_n \in X_n : P_n A x_n = P_n b , \ \|x_n\| \leq \gamma , \ n \in N\} \neq \varnothing .$$

(v) *The equation $Ax = b$ has a unique solution x with $\|x\| \leq \gamma$.*

(vi) $S_n \to \{x\}$.

Proof. (i)

$$\begin{aligned} [Ax - b , Jx] &= [Ax - A(0) , Jx] + [A(0) - b , Jx] \\ &\geq \alpha\|x\|^2 - \|A(0) - b\| \, \|x\| \\ &> \alpha\|x\|^2 - \alpha\gamma\|x\| \geq 0 . \end{aligned}$$

(ii) With no loss of generality, we take $\dim X_n = n$. Let $\{\phi_1 , \dots , \phi_n\}$ be any basis of X_n. Then

$$J x_n = \sum_{i=1}^{n} a_i^n J\phi_i , \qquad x_n \in X_n .$$

The correspondence $x_n \leftrightarrow (a_1^n, \ldots, a_n^n)$ defines an isomorphism $X_n \leftrightarrow R^n$. Let us define $F: R^n \to R^n$ by

$$Fa^n = ([Ax_n - b, J\phi_1], \ldots, [Ax_n - b, J\phi_n]).$$

Then F is continuous, and by (2.3) we have

$$Fa^n = 0 \Leftrightarrow P_n A x_n = P_n b.$$

Next, let $D_n = \{x_n \in X_n : \|x_n\| < \gamma\}$ and $D_n \leftrightarrow D^n \subset R^n$. Then D^n is bounded, open, and convex, and $0 \in D^n$. Also, $\delta D_n \leftrightarrow \delta D^n$. By hypothesis,

$$[Fa^n, a^n] = [Ax_n - b, Jx_n] > 0 \quad \text{for } a^n \in \partial D^n.$$

We obtain, by Lemma 2.2, $Fa^n = 0$ for $a^n \in D^n$. Hence there exists $Jx_n \leftrightarrow a^n$ such that $x_n \in D_n$ and $P_n A x_n = P_n b$.

(iii) Since $P_n \xrightarrow{c} I$ implies $P_n A \xrightarrow{c} A$, it suffices to show that $\{P_n A\}$ is asymptotically regular. To achieve this, let $\{x_n\}$ be bounded and $\{P_n A x_n\}$ d-compact. Then $\{Ax_n\}$ is bounded. Since $P_n A x_n \to b$ for some $b \in X$, $n \in N'' \subset N'$ and, by Lemma 2.3, $J(x_n - x) \xrightarrow{w} 0$ for some $x \in X$, $n \in N' \subset N$, it follows that

$$[Ax, J(x_n - x)] \to 0,$$
$$[Ax_n, P_n^* J(x_n - x)] = [P_n A x_n, J(x_n - x)] \to 0,$$

and

$$[Ax_n - Ax, J(x_n - x)] = [Ax_n, J(x_n - x)] - [Ax, J(x_n - x)] \to 0.$$

Since A is strongly accretive, by condition (2.1), we get

$$\alpha \|x_n - x\|^2 \le [Ax_n - Ax, J(x_n - x)] \to 0.$$

Therefore, $x_n \to x$ for $n \in N''$, and, consequently, $\{x_n\}$ is d-compact and $\{P_n A\}$ asymptotically regular.

(iv) Assertions (i) and (ii) imply that $S_n \neq \varnothing$.

(v) Since A is strongly accretive, this implies that any solution x of $Ax = b$ is unique.

(vi) Since $P_n A \xrightarrow{r} A$ (assertion (iii)) and $P_n b \to b$ (since $P_n \to I$), we obtain from Lemma 2.1 and assertion (v) that $S_n \to \{x\}$. □

3. Application to numerical ranges

This section deals with regular approximation of nonlinear equations involving the numerical ranges of Banach space operators.

We define the numerical range of an operator $A: X \to X$, denoted $n[A]$, by

$$n[A] = \left\{ \frac{[Ax - Ay, J(x - y)]}{\|x - y\|^2} : x, y \in X, \ x \neq y \right\}.$$

The numerical range $n[A]$ has properties similar to those of the Zarantonello numerical range $N[A]$, defined as [17]

$$N[A] = \left\{ \frac{\langle Ax - Ay, x - y \rangle}{\|x - y\|^2} : x, y \in X, \ x \neq y \right\},$$

where X is a Hilbert space and $\langle \cdot, \cdot \rangle$ is the standard inner product on X. Clearly, $n[A]$ reduces to $N[A]$ when X is a Hilbert space. Let us describe some of the elementary properties of $n[A]$.

Theorem 3.1. *Let A, $B: X \to X$ be mappings from a real Banach space X into itself and $\lambda \in K$ (field). Then*

(i) $n[\lambda A] = \lambda n[A]$,
(ii) $n[A+B] \subseteq n[A] + n[B]$, *and*
(iii) $n[A - \lambda I] = n[A] - \{\lambda\}$.

Proof. The proof follows from the definition.

Theorem 3.2. *Let X be a real separable Banach space and its dual X^* uniformly convex. If $A: X \to X$ is bounded and continuous, and a number $\lambda \in K$ is at a positive distance α from the numerical range $n[A]$ of A, that is,*

$$\alpha = \inf\{|\lambda - \mu| : \mu \in n[A]\} > 0,$$

then the following implications hold.

(i) *If $b \in X$, $\|x\| \geq \gamma > \|(A - \lambda I)(0) - b\|/\alpha$ for some $x \in X$, $\gamma > 0$, then $[(A - \lambda I)x - b, Jx] > 0$.*
(ii) *For $b \in X$, $\gamma > 0$, and $[(A - \lambda I)x - b, Jx] > 0$ with $\|x\| = \gamma$, we have $P_n(A - \lambda I)x_n = P_n b$ for some $x_n \in X_n$, $\|x_n\| < \gamma$, $n \in N$.*
(iii) *$P_n(A - \lambda I) \xrightarrow{r} (A - \lambda I)$.*
(iv) *For $b \in X$ and $\gamma > \|(A - \lambda I)(0) - b\|/\alpha$, we have*

$$S_n = \{x_n \in X_n : P_n(A - \lambda I)x_n = P_n b,\ \|x_n\| \leq \gamma,\ n \in N\} \neq \varnothing.$$

(v) *The equation $(A - \lambda I)x = b$ has a unique solution x for $\|x\| \leq \gamma$.*
(vi) *$S_n \to \{x\}$.*

Proof. Since the proof follows from an application of Theorem 2.4, it would suffice to show that $A - \lambda I$ is strongly accretive. For x, $y \in X$, $x \neq y$, we get

$$\begin{aligned}
&|[(A - \lambda I)x - (A - \lambda I)y, J(x - y)]| \\
&\quad = |[Ax - Ay, J(x - y)] - \lambda[x - y, J(x - y)]| \\
&\quad = \left| \frac{[Ax - Ay, J(x - y)]}{\|x - y\|^2} - \lambda \right| \|x - y\|^2 \\
&\quad \geq \alpha \|x - y\|^2.
\end{aligned}$$

Remark 3.3. Theorem 3.2 reduces to the case of the Zarantonello numerical range when X is a Hilbert space.

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