PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

A NEW DUALITY THEOREM FOR SEMISIMPLE MODULES AND CHARACTERIZATION OF VILLAMAYOR RINGS

CARL FAITH AND PERE MENAL

(Communicated by Lance W. Small)

ABSTRACT. We prove the theorem: If R is a ring whose right ideals satisfy the double annihilator condition with respect to a semisimple right R-module W, then every right ideal is an intersection of maximal right ideals, consequently R is a right V (for Villamayor) ring, and W is then necessarily a cogenerator of mod-R. (The converse is well known.) We use this to give a new proof of a theorem of ours on right Johns rings.

INTRODUCTION

A ring R is a *right V ring* provided that R satisfies the f.e.c.'s:

(V1) Every simple right R-module is injective.
(V2) Every right ideal I of R is an intersection of maximal right ideals, equivalently, R/I has zero Jacobson radical, i.e., $\mathrm{rad}(R/I) = 0$.
(V3) Every right R-module M has zero Jacobson radical, i.e., $\mathrm{rad}\, M = 0$.

Proof. See [F1], p. 356, Proposition 7.32A. □

If W is a right R-module, we say that W *satisfies the double annihilator condition* $(=$ *d.a.c.*$)$ *with respect to right ideals* provided that

$$I = \mathrm{ann}_R\, \mathrm{ann}_W I$$

where $\mathrm{ann}_W S$ denotes the annihilator of a subset S of R in W, and dually for $\mathrm{ann}_R S$ for a subset S of W.

A CHARACTERIZATION OF V-RINGS

We now state the new characterization of V-rings.

V-Ring Theorem. *A ring R is a right V-ring iff some semisimple right R-module satisfies the d.a.c. with respect to right ideals.*

Proof. Sufficiency. It suffices to prove that every right ideal of I of R is the intersection of maximal right ideals. Let $M = \mathrm{ann}_W I$. Then by the d.a.c.

$$I = \bigcap_{m \in M} \mathrm{ann}_R m. \tag{1}$$

Received by the editors September 21, 1993.
1991 *Mathematics Subject Classification.* Primary 16D60, 16D10; Secondary 16D50, 16G99.
The second author is deceased (April 4, 1991; see notes in [F-M1] and [P]).

Since mR is a semisimple R-submodule of finite length, then

(2) $$mR = v_1 R \oplus \cdots \oplus v_n R$$

where $v_i \in W$ and $v_i R$ is simple, $i = 1, \ldots, n$. Write

(3) $$m = v_1 a_1 + \cdots + v_n a_n$$

where $a_i \in R$, $i = 1, \ldots, n$; and let

(4) $$H = \bigcap_{i=1}^{n} \operatorname{ann}_R v_i a_i .$$

We assert that

(5) $$H = \operatorname{ann}_R m$$

is the intersection of maximal right ideals. For if $v_i a_i \neq 0$, then $v_i a_i R = v_i R$ is simple, and since

$$v_i R \approx R/(\operatorname{ann}_R v_i a_i),$$

then $\operatorname{ann}_R v_i a_i$ is a maximal right ideal, hence H is the intersection of maximal right ideals.

Next we show that (5) holds. Obviously, $\operatorname{ann}_R m \supseteq H$. To prove the reverse inclusion, note that if $r \in \operatorname{ann}_R m$, then

$$v_1 a_1 r + \cdots + v_n a_n r = 0 .$$

By (2), then $v_i a_i r = 0$, hence $r \in \operatorname{ann}_R v_i a_i$ for all i, that is, $r \in H$, so (5) holds.

Since (4) is the intersection of maximal right ideals, then so is $\operatorname{ann}_R m$, whence I by (1).

Necessity. If R is a right V-ring, then the direct sum W of a complete isomorphic set of simple right R-modules is a minimal cogenerator of mod-R. (See, e.g., [F1], p. 167, Proposition 3.55. There is a misprint in this proposition; cf. [F2].) Furthermore, *every* cogenerator W satisfies the d.a.c. with respect to right ideals. (Hint: if I is a right ideal, then R/I embeds in a direct product W^α of copies of W, say $h : R/I \mapsto W^\alpha$. If $h(1+I) = (w_i) \in W^\alpha$, then $I = \operatorname{ann}_R\{w_i\}_{i\in\alpha}$.) □

Corollary. *If a ring R satisfies the d.a.c. with respect to a semisimple right R-module W, then W is a right cogenerator of R.*

Proof. R is a right V-ring, so every simple right R-module V is injective, so it suffices to show that W contains a copy of each such V. But $V \approx R/M$, where $M \underset{\max}{\triangleleft} R$, and by the d.a.c., $M = \operatorname{ann}_R w$ for some $w \in W$. Since $wR \approx V$, we have $V \hookrightarrow W$ as needed.

F-M Theorem 2.3 ([F-M1]). *If R is a right Johns ring (= right Noetherian and every right ideal is a right annihilator), then R/J is a right V-ring, where J is the Jacobson radical.*

Proof. Let $W = \operatorname{soc} R$. Then

(*) $$J = {}^{\perp}W = W^{\perp}$$

is nilpotent and

(**) $$W = J^{\perp} = {}^{\perp}J$$

where " $\perp$ " denotes an annihilator in R on the appropriate side. (See [F-M], Lemma 2.2.)

By the fact that R_R satisfies the d.a.c. with respect to right ideals, and using $(*)$ and $(**)$, one sees that right ideals of R containing J, hence right ideals of R/J, satisfy the d.a.c. with respect to the semisimple module W. Then by the V-Ring Theorem, we see that R/J is a right V-ring.

Corollary. *If R is such that $W = {}^{\perp}J$ is a semisimple right R-module and every right ideal containing J is a right annihilator, then R/J is a right V-ring.*

Acknowledgment

We wish to acknowledge P. M. Cohn for raising a question about the proof of Theorem 2.3, in connection with his reviewing [F-M1], namely, given a finitely embedded (= f.e.) module M (= has finite essential socle) that occurs in the proof, and given an embedding of M in a direct product S^α of copies of a module ($S = \operatorname{soc} R_R$ in the proof), how does one conclude that M embeds in a finite product S^n of copies of S? This can be resolved as follows. Let $\{p_i\}_{i\in\alpha}$ denote the set of projections $S^\alpha \to S$ of the product S^α. Since $\bigcap_{i\in\alpha} \ker p_i = 0$, and since M is f.e., then for some finite subset $\bar{p}_{i_1}, \dots, \bar{p}_{i_n}$ of the induced maps $\bar{p}_i : M \to S$ we have $\bigcap_{i=1}^n \ker \bar{p}_i = 0$. But then the direct sum of the $\{\bar{p}_i\}_{i=1}^n$ is an embedding $M \to S^n$.

References

[F1] C. Faith, *Algebra* I: *Rings, modules and categories*, Grundlehren Math. Wiss., Bd. 190, Springer-Verlag, Berlin, Heidelberg, and New York, 1973 (corrected reprint, 1981).

[F2] ——, *Minimal cogenerators over Osofsky and Camillo rings*, preprint, 1993.

[F-M1] C. Faith and Pere Menal, *A counter-example to a conjecture of Johns*, Proc. Amer. Math. Soc. **116** (1992), 21–26.

[F-M2] ——, *The structure of Johns rings*, Proc. Amer. Math. Soc. **120** (1994), 1071–1081.

[P] C. Perelló (ed.), *Pere Menal memorial volumes*, Publ. Mat. **36** (1992).

Department of Mathematics, Rutgers University New Brunswick, New Jersey 08903

Permanent address: 199 Longview Drive, Princeton, New Jersey 08540