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SKEW POLYNOMIAL EXTENSIONS OF COMMUTATIVE NOETHERIAN JACOBSON RINGS

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ABSTRACT. The Jacobson condition (i.e., that all prime ideals are semiprimitive) is proved to pass from a commutative noetherian ring R to a skew polynomial ring $R[y; \tau, \delta]$, assuming only that τ is an automorphism.

1. INTRODUCTION

This note is concerned with the prime ideal structure of a skew polynomial ring $S = R[y; \tau, \delta]$ over a noetherian ring R with respect to an automorphism τ and a (left) τ-derivation δ (cf. [7]). An unanswered question in this setting is whether S must satisfy the Jacobson condition (i.e., every prime ideal is an intersection of primitive ideals) when R satisfies the same property. Some positive answers are known even for non-noetherian coefficient rings: Watters **[15]** proved that $K[y]$ is Jacobson for any Jacobson ring K, and Irving **[9]** showed that an iterated skew polynomial extension T of a commutative Jacobson ring K is Jacobson if K is central in T (see also **[12]**). On the other hand, examples have been constructed of non-noetherian commutative Jacobson rings K with skew polynomial extensions $K[y; \tau, \delta]$ that are not Jacobson; see Pearson and Stephenson **[14]** for an example in which $\delta = 0$, and see Bergen, Montgomery, and Passman **[1]** or Ferrero and Kishimoto **[3]** for examples in which $\tau = 1$. Within the noetherian context, affirmative answers to the problem were given by Goldie and Michler **[4]** when δ is trivial, and by Jordan **[10]** when τ is the identity.

The aim of this note is to provide an affirmative answer to the above question when R is commutative noetherian but no restrictions are placed upon τ or δ. Such a result has remained unavailable despite the thorough analyses of the commutative case by Irving **[8]** and the first author **[5]**. Our methods rely in part on the techniques introduced in **[6]** as well as on the results in **[5]**. Moreover, it is not assumed that R be filtered, graded, or affine.

We impose the blanket hypotheses throughout that R is a commutative noetherian ring, that $S = R[y; \tau, \delta]$, and that τ is an automorphism of R. However, commutativity of R is not needed for (2.2) and (3.1).

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2. Induced vs. noninduced prime ideals

Throughout this section we let P denote an arbitrary prime ideal of S. If A is a ring and I is an ideal of A, then $N(I)$ denotes the intersection of all the prime ideals containing I and $J(I)$ the intersection of all the right primitive ideals containing I. The reader is referred to [7, **13**] for further explanations of undefined terms.

2.1. By [**6**, 5.3, 5.5], we may fix a prime ideal Q of R, minimal over $P \cap R$, that satisfies the following property: If A denotes the Goldie quotient ring of R/Q, then P is the right annihilator in S of a nonzero A-S-bimodule factor M of $A \otimes_R S$ such that $M_{S/P}$ is torsionfree. Next, set $U = S/QS$, and let e denote the coset $1 + QS$. Observe that we may identify ${}_R U_S$ with $(R/Q) \otimes_R S$ by an isomorphism that sends e to $1 \otimes 1$, and under this identification we may view ${}_{(R/Q)}U$ as a free left (R/Q)-module with basis

$$\{1 \otimes 1,\ 1 \otimes y,\ 1 \otimes y^2, \ldots\}.$$

Also, observe that as a left R-module, $A \otimes_R S$ is isomorphic to an Ore localization of ${}_R U$.

It follows from the above choice of Q that $\operatorname{ann} U_S \subseteq P$, since $\operatorname{ann} U_S = \operatorname{ann}(A \otimes_R S)_S$. Our analysis divides into the two cases determined by whether or not $P = \operatorname{ann} U_S$, and we begin with an incomparabilty result.

2.2. **Lemma.** *(Here R need not be commutative.) Suppose that J is an ideal of S properly containing P. If $P \neq \operatorname{ann} U_S$, then $J \cap R \not\subseteq Q$.*

Proof. By [**6**, 4.6], the set $\mathscr{C}$ of regular elements of $R/(P \cap R)$ forms an Ore set (of regular elements of S/P) in both $R/(P \cap R)$ and S/P, and the ring $E = (R/(P \cap R))\mathscr{C}^{-1}$ is artinian. Letting $F = (S/P)\mathscr{C}^{-1}$, we see that the canonical embedding of $R/(P \cap R)$ into S/P extends uniquely to an embedding of E into F. Now choose an ideal I of S that contains P and is maximal among those ideals of S whose intersection with R lies within Q. Standard arguments reveal that I is a prime ideal of S disjoint from $\mathscr{C}$. Consequently, if I strictly contains P, then I extends to a proper nonzero ideal of F (e.g., [7, 9.22]). Next, it follows from [**6**, 5.7, 5.8] that F_E is finitely generated when $P \neq \operatorname{ann} U_S$. However, if F has finite length as a right E-module, then F is a simple artinian ring. Therefore, $I = P$ and the lemma follows. □

2.3. **Lemma.** $(P + QS) \cap R = Q$.

Proof. We may assume without loss of generality that $P \cap R \neq Q$, and it therefore follows from the minimality of Q that $P \cap R$ is not prime. Moreover, it suffices to prove that $(P + QS) \cap R \subseteq Q$. Next, by [**5**, 3.1], either $P \cap R$ is semiprime or $R/(P \cap R)$ has a unique associated prime. We first consider the case where $R/(P \cap R)$ is semiprime, and we let $Q, Q_2, \ldots, Q_n$ be the distinct prime ideals of R minimal over $P \cap R$. Note that $n \geq 2$ and $Q_n Q_{n-1} \cdots Q_2 Q \subseteq P \cap R$. Hence,

$$Q_n Q_{n-1} \cdots Q_2 [(P + QS) \cap R] \subseteq P \cap R \subseteq Q.$$

Since $Q_n Q_{n-1} \cdots Q_2 \not\subseteq Q$, it follows that $(P + QS) \cap R \subseteq Q$ in this case.

Now assume that $R/(P \cap R)$ has a unique associated prime. Consequently, Q is the unique prime ideal of R minimal over $P \cap R$ and $\mathscr{C}_R(Q) \subseteq \mathscr{C}_R(P \cap R)$. Therefore, $\mathscr{C}_R(Q) \subseteq \mathscr{C}_S(P)$ by [**6**, 4.6]. Hence, if there exists an element $c \in (P + QS) \cap (R \setminus Q)$, then $c \in \mathscr{C}_S(P)$. Next observe that there exists a positive integer n such that $Q^n \subseteq P \cap R$ while $Q^{n-1} \not\subseteq P \cap R$. However, it now follows that $Q^{n-1}c \subseteq Q^{n-1}(P + QS) \subseteq P$, in contradiction to the regularity of c modulo P. Therefore, $(P + QS) \cap R \subseteq Q$ and the lemma follows. $\square$

The proof of the following proposition is adapted from [**4**, **10**].

2.4. **Proposition.** *If $P \neq \operatorname{ann} U_S$ and Q is semiprimitive, then P is semiprimitive.*

Proof. For $t = 0, 1, \dots$ set

$$\begin{aligned} K_t &= \{a \in R \mid e.(ay^t + a_{t-1}y^{t-1} + \cdots + a_0) \in UP \text{ for some } a_0, \dots, a_{t-1} \in R\} \\ &= \{a \in R \mid ay^t + a_{t-1}y^{t-1} + \cdots + a_0 \in P + QS \text{ for some } a_0, \dots, a_{t-1} \in R\}. \end{aligned}$$

Then let $K = K_n$, where n is the minimum value for t such that

$$0 \neq e.(a_t y^t + a_{t-1}y^{t-1} + \cdots + a_0) \in UP$$

for some $a_0, \dots, a_t \in R$. (The existence of n follows from the assumption that $P \neq \operatorname{ann} U_S$.) Note, since τ is an automorphism, that K is an ideal of R containing Q, and observe, for $a \in K$, that $a \notin Q$ if and only if $0 \neq e.(ay^n + a_{n-1}y^{n-1} + \cdots + a_0) \in UP$ for some $a_0, \dots, a_{n-1} \in R$. In particular, K properly contains Q. Moreover, since $(P + QS) \cap R \subseteq Q$ by (2.3), it follows that $n \geq 1$.

Now let M be a maximal ideal of R that contains Q. We claim that either $J(P) \cap R \subseteq M$ or $K \subseteq M$. To prove this claim, assume that $J(P) \cap R \not\subseteq M$. Choose $j \in J(P) \cap R$ such that $j \notin M$. There then exist $m \in M$ and $b \in R$ such that $1 = m + jb$. Since $jb \in J(P)$, there exists a polynomial $f = cy^\ell + c_{\ell-1}y^{\ell-1} + \cdots + c_0 \in S$, with $c, c_0, \dots, c_{\ell-1} \in R$ and $c \neq 0$, such that $(1 - jb)f = mf \equiv 1 \pmod{P}$. Hence, $e.mf \equiv e \pmod{UP}$. Now choose $a \in K \setminus Q$. There then exists a polynomial $p = ay^n + a_{n-1}y^{n-1} + \cdots + a_0 \in S$, with $a_0, \dots, a_{n-1} \in R$, for which $0 \neq e.p \in UP$. Assume for the moment that $\ell \geq n$, and observe that

$$af - p\tau^{-n}(c)y^{\ell-n}$$

has degree less than ℓ. It now follows from a straightforward induction that $e.a^k f \equiv e.r \pmod{UP}$ for some nonnegative integer k and some polynomial $r \in S$ with degree $d < n$. Hence, we have

$$e.a^k mf = m.e.a^k f \equiv m.e.r = e.mr \pmod{UP},$$

and since $a^k mf \equiv a^k \pmod{P}$, we see that $e.a^k \equiv e.mr \pmod{UP}$. Consequently, $e.(a^k - mr) \in UP$. However, $a^k - mr$ has degree strictly less than n. Therefore, it follows from the choice of n that $e.(a^k - mr) = 0$. Hence, $a^k - mr_0 \in Q$, where r_0 is the constant term of r. But this last statement implies that $a^k \in M$, because $Q \subseteq M$. Thus $a \in M$, and it therefore follows from the choice of a that $K \subseteq M$. This verifies the claim. Furthermore, it follows from the claim that $J(P) \cap K \subseteq M$. Because M was an arbitrary maximal ideal of R containing Q, we now see that $J(P) \cap K \subseteq J(Q) = Q$.

But this inclusion means that $J(P) \cap R \subseteq Q$, since $K \not\subseteq Q$. Thus by (2.2), $J(P) = P$, and the lemma is proved. □

2.5. **Lemma.** *Assume that* $P = \operatorname{ann} U_S$. *Then* $P \cap R$ *is* (τ, δ)*-prime, and* $P = (P \cap R)S = S(P \cap R)$. *Consequently, if* τ *and* δ *also denote their induced actions on* $R/(P \cap R)$, *and* y *also denotes its image in* S/P, *then* $S/P = (R/(P \cap R))[y; \tau, \delta]$.

Proof. Set $I = P \cap R$. It follows from [**6**, 5.9ii] that there exists an $n \in \mathbb{N}$ such that $\tau^n(Q) = Q$ and such that $\{Q, \tau(Q), \ldots, \tau^{n-1}(Q)\}$ is the set of prime ideals of R minimal over $P \cap R$. In particular, $N(I)$ is τ-stable. Now suppose that $I = Q$. Then I is τ-stable and therefore (τ, δ)-stable (e.g., [**6**, 2.1v]). Hence, $IS = SI$, and $P = \operatorname{ann}(S/IS)_S = IS$. Further, it is a triviality that I is (τ, δ)-prime. Next, assume that $I \neq Q$. Consequently, I is not a prime ideal, and so I is a (τ, δ)-prime ideal by [**5**, 3.1]. It therefore follows from [**5**, 3.3] that $P_0 = IS = SI$ is a prime ideal of S. Moreover, $P_0 \subseteq P$ and $P_0 \cap R = P \cap R = I$.

Because Q is minimal over I, and R is commutative, it follows that Q is an annihilator prime of $(R/I)_R$. In particular, Q is an annihilator prime of $(S/P_0)_R$. Hence, by [**6**, 5.5], $P_0 \supseteq \operatorname{ann} U_S = P$. The lemma follows. □

2.6. **Lemma.** *Suppose that* Q *is a maximal ideal of* R *and that* S/P *is artinian. Then* S/P *has finite length as a right* R*-module.*

Proof. First, it follows from [**6**, 4.4] that every prime ideal of R minimal over $P \cap R$ is maximal, and so $R/(P \cap R)$ is artinian. Therefore, if $P \neq \operatorname{ann} U_S$, the desired conclusion follows from [**6**, 5.9i]. Now assume that $P = \operatorname{ann} U_S$. Therefore, by (2.5), we may assume without loss of generality that $P = 0$. But then y is a regular noninvertible element of S, a contradiction to the fact that S is artinian (e.g., [**13**, 3.1.1]). □

3. Induced bimodules

Chapter 5 of [**6**] contains an extensive analysis of the prime ideals of S that occur as annihilators of factors of bimodules of the form $A \otimes_R S$ where A is the Goldie quotient ring of a prime factor ring of R. We shall need one element of the corresponding analysis of bimodule *sub*factors of $A \otimes_R S$, as follows. In the case of a bimodule factor, this lemma is a consequence of [**6**, 5.4, 5.5].

3.1. **Lemma.** (*Here* R *need not be commutative.*) *Let* P *be a prime ideal of* S *and* Q *a prime ideal of* R, *and let* A *denote the Goldie quotient ring of* R/Q. *Further assume that* P *is the right annihilator in* S *of an* A-S*-bimodule subfactor* K *of* $A \otimes_R S$ *that is torsionfree as a right* (S/P)*-module. Then every prime ideal in* R *minimal over* $P \cap R$ *belongs to the* τ*-orbit of* Q.

Proof. Choose a nonzero element $\ell \in K$ and let $L = A.\ell.R$. It follows from [**6**, 4.6] that $R/(P \cap R)$ has an artinian quotient ring and that every regular element of $R/(P \cap R)$ is regular in S/P. Hence, L is torsionfree as a right $(R/(P \cap R))$-module, and by Small's Theorem (e.g., [**7**, 10.10]) and [**7**, 6.3], it follows that every annihilator prime of L_R is minimal over $P \cap R$. We leave to the reader the verification that L has finite length as a left A-module. Now choose a simple A-R-sub-bimodule M of L. The right annihilator in R of M

is a prime ideal, say Q', and we have just seen that Q' must be minimal over $P \cap R$. However, it follows from the proof in [**6**, 4.4] that M is isomorphic to A^{τ^j} as an A-R-bimodule, for some positive integer j. (As a left A-module, A^{τ^j} has the same structure as A, but the right R-module structure is defined by the operation $a * r = a\tau^j(r)$, for every $r \in R$ and $a \in A$.) It therefore follows that $Q' = \tau^{-j}(Q)$, and the desired conclusion now follows from [**6**, 4.4]. □

3.2. **Proposition.** *Let M be a maximal ideal of R. Then the right annihilator in S of S/MS is prime.*

Proof. Set $V = S/MS = (R/M) \otimes_R S$, and let P denote a maximal annihilator prime of V_S. It follows from [**6**, 5.6iv] that V is uniform as an R-S-bimodule, and it is therefore easy to verify that every annihilator prime of V_S is contained in P. If $P = \operatorname{ann} V_S$, then there is nothing to prove, and so we suppose otherwise. Next, let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be an affiliated series for V (see, e.g., [7, p. 33]), where $n > 1$, and set $P_i = \operatorname{ann}(V_i/V_{i-1})_S$ for $1 \leq i \leq n$. (Note that $P_1 = P$.) If $i > 1$, it follows from [**6**, 5.6iii] that V_i/V_{i-1} has finite length as a left R-module. It therefore can be deduced from Lenagan's Theorem (e.g., [7, 7.10]) that $(V_i/V_{i-1})_S$ has finite length for $i > 1$. However, it now follows from [7, 7.2] that S/P_i is an artinian ring. In particular, each V_i/V_{i-1} is torsionfree as a right (S/P_i)-module, and in view of (3.1), the prime ideals of R minimal over $P \cap R$ are therefore maximal ideals. We may now conclude from (2.6) that each S/P_i has finite length as a right R-module for $i > 1$.

We next prove that $S/P = S/P_1$ is artinian and has finite length as a right R-module. If P_2 is an annihilator prime of V_S, then $P_2 \subseteq P$ and there is nothing to prove. So we may assume otherwise. It then follows from [**11**, 1.2] that there is a series of links (e.g., [7, p. 178]) from P_2 to some annihilator prime P' of V_S. However, it now follows from [7, 7.2, 7.10] that P' is coartinian. Hence $P = P'$ is coartinian, because $P' \subseteq P$. Next, it follows from (3.1) that every prime ideal of R minimal over $P \cap R$ is a maximal ideal. Thus S/P has finite length as a right R-module by (2.6).

To conclude, it now follows that V_i/V_{i-1} has finite length as a right R-module for all $1 \leq i \leq n$. But we are now forced to conclude that V_R has finite length, an absurdity. The lemma follows. □

4. Ascendancy of the Jacobson condition

4.1. **Lemma.** *Assume that R is artinian and (τ, δ)-prime. Then S is a Jacobson ring.*

Proof. First, it follows from [**5**, 2.3] and [**4**, 5*] that R is (τ, δ)-simple. Also, R is a Jacobson ring, and so by [**10**, 3.5] we may assume that τ is not the identity. Now assume that R is τ-prime. Then it follows from [**5**, 3.7] that δ is inner, and so the desired conclusion follows from [**4**, 1.11*] and, for example, [**5**, 1.5c]. It remains to consider the case that R is not τ-prime. Therefore, by [**5**, 2.6], R is δ-prime and has a unique maximal ideal M. From [**5**, 2.6, 4.6] it follows that S contains a subring $A = (R/M)[y'; \delta']$, where $y' \in S$ and δ' is a derivation of R/M, and it follows from [**10**, 3.5] that A is a Jacobson ring. It is proved in [**5**, 4.6] that S is finitely generated as a left A-module. Therefore, S is a Jacobson ring by [**2**, Theorem 1]. □

Recall that a prime ideal P of S is said to *lie over* a prime ideal Q of R when Q is minimal over $P \cap R$.

4.2. **Lemma.** *Assume that there exists a maximal ideal M of R such that the module $V = (S/MS)_S$ is faithful. Then S is semiprimitive.*

Proof. First suppose that M is minimal. By (3.2), S is prime, and so by [**6**, 5.12], the minimal prime ideals of R are all contained within a single τ-orbit. Therefore, all minimal prime ideals of R are maximal, and so R is artinian. Moreover, because S is prime, and because nonzero (τ, δ)-ideals of R induce to nonzero ideals of S, it follows that R is (τ, δ)-prime. Hence, by (4.1), S is semiprimitive. Thus we may assume that M is not minimal.

Next, suppose that $\tau(M) = M$. Since V_S is faithful, MS cannot be an ideal of S, and so M is not δ-stable. Thus no ideal of S contracts to M; see [**6**, 2.1v]. Now suppose that N is a prime ideal of S lying over M. From the preceding observation it follows that $N \cap R \neq M$, and so $I = N \cap R$ must be a (τ, δ)-prime ideal of R by [**5**, 3.1]. Moreover, our assumption that M not be a minimal prime ideal of R guarantees that $I \neq 0$. Hence IS is a nonzero ideal of S contained in MS, a contradiction to the faithfulness of V_S. Thus, no prime ideal of S lies over M. It therefore follows from [**6**, 5.7] that there exist no proper simple R-S-bimodule factors of V, and so ${}_RV_S$ is a simple bimodule. It is now straightforward to prove as follows that S is right primitive: Let K be a maximal right S-submodule of V, and let $J = \operatorname{ann}(V/K)_S$. Then VJ is an R-S-sub-bimodule of V that is not equal to V. Hence $VJ = 0$, and so $J = 0$ by the faithfulness of V_S. Therefore V/K is a faithful simple right S-module.

Finally, assume that $\tau(M) \neq M$. Let $L = \bigcap_{i \in \mathbb{Z}} \tau^i(M)$, and note that L is a semiprime, τ-prime ideal. By [**5**, 3.1], for each $i \in \mathbb{Z}$ there exists a prime ideal of S contracting to $\tau^i(M)$. Hence, there exists an ideal of S contracting to L, and it follows, for example, from [**6**, 2.1v] that L is (τ, δ)-stable. Therefore, $LS = SL$ is an ideal of S contained within MS, and so $LS = 0$ because V_S is faithful. Consequently, $L = 0$, and hence R is a semiprime, τ-prime ring.

To conclude, let $J = J(S)$, and suppose that $J \neq 0$. Note that the set of leading coefficients of elements of J, together with 0, namely the set

$$\{a \in R \mid ay^t + a_{t-1}y^{t-1} + \cdots + a_0 \in J \text{ for some } a_0, \ldots, a_{t-1} \in R\},$$

is a nonzero τ-ideal of R. This ideal must contain a regular element since R is τ-prime, and therefore there exists a polynomial $f \in J$ with positive degree and regular leading coefficient. Since $1+f$ is a unit, there exists another polynomial g such that $(1+f)g = 1$. But the degree of $(1+f)g$ is certainly greater than zero, by the regularity of the leading coefficient of f, and we thus obtain a contradiction. Hence, $J = 0$, and the lemma follows. □

4.3. **Theorem.** *Assume that R is a commutative noetherian Jacobson ring. Then the skew polynomial ring $S = R[y; \tau, \delta]$ is a Jacobson ring.*

Proof. Suppose that the theorem is false, and let P denote a maximally chosen nonsemiprimitive prime ideal of S. As in (2.1), we may select a prime ideal Q of R such that Q is minimal over $P \cap R$ and such that P is the annihilator in S of an A-S-bimodule factor of $A \otimes_R S$, where A is the field of fractions for R/Q. If $P \neq \operatorname{ann}(S/QS)_S$, then P is semiprimitive, by (2.4). Therefore, by (2.5), we may assume without loss of generality that $P = 0$. Furthermore, Q is equal to the intersection of those maximal ideals of R that contain it. In particular,

$$QS = \bigcap \{ MS \mid M \in \max R \text{ and } M \supseteq Q \}.$$

Therefore,

$$0 = \operatorname{ann}(S/QS)_S = \bigcap \{ \operatorname{ann}(S/MS)_S \mid M \in \max R \text{ and } M \supseteq Q \}.$$

Next, it follows from the above equalities and (3.2) that if there exists no maximal ideal M in S such that $M \supseteq Q$ and $(S/MS)_S$ is faithful, then some intersection of nonzero prime ideals in S is equal to zero, a contradiction to the fact that each nonzero prime ideal of S is semiprimitive. Thus, there exists a maximal ideal M in R such that $(S/MS)_S$ is faithful. Therefore, it follows from (4.2) that S is semiprimitive, a contradiction to our hypothesis. The theorem follows. □

4.4. **A question of Small.** A possible generalization of the preceding theorem would include the replacement of the commutativity hypothesis with the assumption that R satisfy a polynomial identity. L. W. Small has informed us of his unpublished proof that if R is an affine PI algebra over a (τ, δ)-constant field k, then $S[u, v] = R[y; \tau, \delta][u][v]$ is generically flat over $k[u]$, and consequently, S is a Jacobson ring (cf. [**13**, 9.3.13]). Small further raises the following question: If T is a filtered noetherian ring such that $\operatorname{gr} T$ is Jacobson, must T also be Jacobson? (We thank L. W. Small for the remarks discussed here.)

Note added in proof (December 1994)

A. D. Bell has communicated two counterexamples to Small's question; however, in one example the filtration is a $\mathbb{Z}$-filtration, while in the other, $\operatorname{gr} T$ is not noetherian. The following modification of Small's question remains open: If T is a nonnegatively filtered noetherian ring such that $\operatorname{gr} T$ is Jacobson and noetherian, must T be noetherian?

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