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FREE INVOLUTIONS ON E_{4m} LATTICES

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ABSTRACT. We determine all the conjugacy classes of traceless involutions on the E_{4m} lattices. In particular, we show that for every $m > 2$ there exist precisely two nonconjugate involutions which induce free $\mathbf{Z}[\mathbf{Z}_2]$-module structures. By inspecting the parity of the E_{4m} form twisted by any such involution, we deduce that a closed, simply connected, topological 4-manifold with intersection form E_{4m} supports a locally linear involution if and only if m is odd and the Kirby-Siebenmann invariant of the manifold is trivial.

1. E_{4m} LATTICES

Recall that a *lattice* in $\mathbf{R}^n$ is a subgroup $L \subset \mathbf{R}^n$ which is additively generated by some basis $b_1, \dots, b_n$ of $\mathbf{R}^n$. The *volume* of the quotient torus $\mathbf{R}^n/L$ can be defined by

$$\operatorname{vol}(\mathbf{R}^n/L) = |\det(b_1, \dots, b_n)|.$$

A lattice is called *unimodular* if $\operatorname{vol}(\mathbf{R}^n/L) = 1$. If $L_0 \subset L$ is a sublattice of L of (necessarily finite) index $|L/L_0|$, then

$$\operatorname{vol}(\mathbf{R}^n/L_0) = \operatorname{vol}(\mathbf{R}^n/L)|L/L_0|.$$

This provides a convenient way of testing whether a set of vectors $v_1, \dots, v_n \in L$ is a basis, the sufficient and necessary condition being

$$|\det(b_1, \dots, b_n)| = |\det(v_1, \dots, v_n)|.$$

Let $e_1, \dots, e_{4m}$ denote an orthonormal basis in $\mathbf{R}^{4m}$ with respect to the usual inner product. The vectors e_i+e_j and $\frac{1}{2}(e_1+\cdots+e_{4m})$ span a lattice $E_{4m} \subset \mathbf{R}^{4m}$ which is a positive definite inner product space over $\mathbf{Z}$ with respect to the restriction of the standard inner product; cf. Milnor-Husemoller [10, II.6.1]. More explicitly,

$$E_{4m} = \{\xi_1 e_1 + \cdots + \xi_{4m} e_{4m} \in \mathbf{R}^{4m} \colon 2\xi_i \in \mathbf{Z},\ 2\xi_1 \equiv \cdots \equiv 2\xi_{4m} \pmod 2,\\ \xi_1 + \cdots + \xi_{4m} \equiv 0 \pmod 2\}.$$

2. $\mathbf{Z}$-TORSION-FREE MODULES OVER $\mathbf{Z}[\mathbf{Z}_2]$

Let L be a free $\mathbf{Z}$-module, and let $T\colon L \to L$ denote an involution on L. Put $\Lambda = \mathbf{Z}[T]$. Thus L becomes a Λ-module. By Reiner's classification

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of $\mathbf{Z}$-torsion-free Λ-modules [1, 74.3], L can be written as the direct sum $L \cong m\mathbf{Z}_+ \oplus n\mathbf{Z}_- \oplus r\Lambda$, where $\mathbf{Z}_\pm$ are copies of $\mathbf{Z}$ on which T acts by $\pm\mathrm{id}$, respectively. Then the triple (m, n, r) determines the isomorphism type of L. Note that Reiner's classification is quite elementary in the case of $\mathbf{Z}[\mathbf{Z}_2]$-modules.

3. Traceless involutions on E_{4m} lattices

In this section we discuss involutions on the E_{4m} lattices $(m > 2)$ for which

$$E_{4m} \cong m\mathbf{Z}_+ \oplus m\mathbf{Z}_- \oplus r\Lambda.$$

Let D_{4m} denote the lattice generated by the vectors $\pm e_i \pm e_j$, $i \neq j$. By Serre [12, p. 40], the isometry group of D_{4m} consists of permutations and sign changes of the vectors e_i. Let T be an involution on E_{4m}, and suppose $m > 2$. Then T preserves D_{4m}, since $\pm e_i \pm e_j$ are precisely the minimal vectors in E_{4m} of norm 2. It follows that the isometry group of E_{4m} consists precisely of permutations and sign changes of an even number of the vectors e_i.

Therefore we may assume that with respect to our orthornormal basis $\{e_i\}$ of $\mathbf{R}^{4m}$, the matrix of T is of the form

$$\begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_{2m} \end{bmatrix},$$

where each Jordan block B_i is one of

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We would ultimately like to know if E_{4m} supports a *free* Λ-module structure. This leads us to the analysis of *traceless* involutions, i.e. those for which the number of $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$-blocks is equal to the number of $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$-blocks. For every such involution T we determine the corresponding Λ-module structure for E_{4m}, by providing an explicit T-invariant basis.

Note that we do not need to consider the $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$-blocks, except when all the remaining blocks are $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Indeed, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

the $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$-blocks can be removed by conjugation.

Therefore we only need to consider the following involutions:

(1) $T_x = p\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \oplus p\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \oplus 2(m-p)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $1 \le p \le m$,

(2) $T_y = \left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right] \oplus (2m-1)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,

(3) $T_z = 2m\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

In order to determine the $\mathbf{Z}[\mathbf{Z}_2]$-module structure induced on the E_{4m} lattice we explicitly construct equivariant bases in each case as follows.[1]

[1] To choose a basis of an n-dimensional vector space over $\mathbf{R}$, it suffices to pick *randomly* n vectors from the space. We extended this "algorithm" to our situation: vectors $v_1, \ldots, v_{4m}$ are chosen

Let $1 \le p \le m$. We define a basis $x_1, \dots, x_{4m}$ by

$$x_i = \begin{cases} \frac{1}{2}\sum_1^{4m} e_j, & i = 1, \\ \frac{1}{2}(\sum_1^{2p} e_j - \sum_{2p+1}^{4p} e_j + \sum_{4p+1}^{4m} e_j), & i = 2, \\ e_{i-1} - e_i, & i = 3, \dots, 4p-1, \\ e_{2p} + e_{2p+1}, & i = 4p, \\ e_i + e_{i+2}, & p < m,\ i = 4p+1, \dots, 4m-2, \\ e_1 + e_{4p+1}, & p < m,\ i = 4m-1, \\ e_1 + e_{4p+2}, & p < m,\ i = 4m. \end{cases}$$

Let $X_{4p,4m}$ denote the matrix of coefficients of the vectors $x_1, \dots, x_{4m}$ with respect to the orthonormal basis.

Lemma 3.1. *For every m and p, such that $1 \le p \le m$, $\det X_{4p,4m} = -1$. The basis $\{x_i\}_{i=1}^{4m}$ is T_x-invariant and*

$$E_{4m} \cong 2(p-1)\mathbf{Z}_+ \oplus 2(p-1)\mathbf{Z}_- \oplus 2(m-p+1)\mathbf{Z}[T_x]. \quad \square$$

We define $y_1, \dots, y_{4m}$ by

$$y_i = \begin{cases} \frac{1}{2}\sum_1^{4m} e_j, & i = 1, \\ \frac{1}{2}\sum_1^{4m} e_j - e_1 - e_2, & i = 2, \\ -e_1 + e_3, & i = 3, \\ e_{i-2} + e_i, & 4 \le i \le 4m. \end{cases}$$

Let Y_{4m} denote the matrix of coefficients of the vectors $y_1, \dots, y_{4m}$ with respect to the orthonormal basis.

Lemma 3.2. *For every $m \ge 1$, $\det Y_{4m} = 1$. The basis $\{y_i\}_{i=1}^{4m}$ is T_y-invariant and $E_{4m} \cong 2m\mathbf{Z}[T_y]$.* $\square$

Finally, we define the basis $z_1, \dots, z_{4m}$ by

$$z_i = \begin{cases} \frac{1}{2}\sum_1^{4m} e_j, & i = 1, \\ e_1 + e_2, & i = 2, \\ e_1 - e_2, & i = 3, \\ e_3 - e_4, & i = 4, \\ e_{i-2} + e_i, & i = 5, \dots, 4m. \end{cases}$$

Let Z_{4m} denote the matrix of coefficients of the vectors $z_1, \dots, z_{4m}$ with respect to the orthonormal basis.

Lemma 3.3. *For every $m \ge 1$, $\det Z_{4m} = -1$. The basis $\{z_i\}_{i=1}^{4m}$ is T_z-invariant and $E_{4m} \cong 2\mathbf{Z}_+ \oplus 2\mathbf{Z}_- \oplus 2(m-1)\mathbf{Z}[T_z]$.* $\square$

We summarize our results in the following theorem.

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randomly from a set of minimal vectors until two conditions are satisfied: $\det(v_1, \dots, v_{4m}) = \pm 1$ and $\{v_1, \dots, v_{4m}\} = \{Tv_1, \dots, Tv_{4m}\}$. We used a set-theoretic, interpreted extension of the C programming language [8] to implement this algorithm.

Theorem 3.4. *For every $m > 2$, there exist precisely two nonconjugate involutions on the E_{4m} lattice which induce free $\mathbf{Z}[\mathbf{Z}_2]$-module structures, namely T_x (when $p = 1$) and T_y. For these involutions, the twisted form, given by*

$$(x, y) \mapsto (x, Ty),$$

is even if and only if $T = T_y$ and m is odd.

4. Application

In this section we use Theorem 3.4 to prove a nonexistence result for involutions on the E_{4m} manifolds.

Let M^4 be an oriented, closed, simply connected, topological 4-manifold. Since M is simply connected, $L = H^2(M; \mathbf{Z})$ is torsion-free. The *intersection form* $\lambda_M: L \times L \to \mathbf{Z}$ of M is defined by

$$\lambda_M(x, y) = (x \cup y)[M].$$

The form λ_M is symmetric and bilinear. Since M is closed, λ_M is unimodular, by Poincaré duality. For example, $\lambda_{S^2 \times S^2} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\lambda_{\pm \mathbf{C}P^2} = [\pm 1]$.

According to the classification theorem of Freedman [7] the intersection form and the Kirby-Siebenmann obstruction $\mathrm{KS}(M) \in H^4(M; \mathbf{Z}_2) = \mathbf{Z}_2$ completely determine the homeomorphism type of M. Note that if λ is even, then the Kirby-Siebenmann obstruction is determined by the signature of λ: $\mathrm{KS}(M) \equiv \frac{1}{8}\sigma(M) \pmod 2$. On the other hand, *every* symmetric, bilinear, unimodular form λ can be realized as the intersection form of a compact 4-manifold. If λ is even, there is a unique M realizing λ. Otherwise, there are two (homotopy equivalent) 4-manifolds realizing the two possible values for the Kirby-Siebenmann obstruction.

Additional structures on M limit the set of forms which can be realized. For instance, the intersection forms of simply connected, closed, *spin* 4-manifolds are precisely the *even* forms. A theorem of Rohlin [11] implies that if M is *smooth* and *spin*, then $\sigma(M) \equiv 0 \pmod{16}$. Thus many topological 4-manifolds cannot be smoothed, e.g., the E_8 manifold. Donaldson [2, 3] showed that the only definite forms realized as the intersection forms of *smooth*, compact 4-manifolds are the standard diagonalizable forms. For instance, his theorem implies that the E_{16} manifold is exotic. Note that this manifold was not detected by Rohlin's theorem. Kwasik and Vogel [9] proved that if a topological 4-manifold M supports a *locally linear* involution, then $\mathrm{KS}(M)$ is trivial. For instance, the E_8 manifold does not admit a locally linear involution. Below we consider all the E_{4m} manifolds which are not detected by the theorem of Kwasik and Vogel in the case when m is divisible by 4.

Theorem 4.1. *Let M denote an E_{4m} manifold. Then M admits a locally linear involution if and only if m is odd and the Kirby-Siebenmann obstruction of M vanishes.*

Proof. Suppose m is odd and $\mathrm{KS}(M) = 0$. By (3.4), $H_2(M)$ supports a free involution which preserves the intersection pairing and induces an even twisted form. By Edmonds-Ewing [6, 7.1], M admits a locally linear involution with two fixed points.

Now suppose that either m is even or $\mathrm{KS}(M) \neq 0$. If $\mathrm{KS}(M) \neq 0$, then M admits no locally linear involution, by Kwasik-Vogel [9]. We may, therefore, assume that $m \equiv 0 \pmod 4$. Let $g: M \to M$ be a locally linear involution. By Edmonds [5, 4.1], the fixed-point set F consists either of precisely two points or of a single, nullhomologous 2-sphere, and $H_2(M)$ is free over $\mathbf{Z}[\mathbf{Z}_2]$. A direct geometric argument implies that $x \cdot g_* y = x \cdot [F] \pmod 2$. It therefore follows that the twisted form $(x, y) \mapsto x \cdot g_* y$ is even. By (3.4), every free involution on $H_2(M)$ induces an odd twisted form. The result follows. □

Remark 4.2. One can show that every simply connected, closed, *indefinite* 4-manifold with trivial Kirby-Siebenmann obstruction admits a locally linear involution. The result of Donaldson [2] implies that every *smooth* simply connected 4-manifold admits a locally linear involution. Edmonds conjectured in [4] that every simply connected, closed, *odd* 4-manifold M also supports such an action, provided $\mathrm{KS}(M) = 0$. The E_{16m} manifolds are the first explicit examples of four-manifolds whose Kirby-Siebenmann invariants are trivial and which do no admit locally linear involutions.

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